Symmetry: Art and Science
Tihany Congress, 2004

DEVELOPMENT OF THE FUNDAMENTAL THEOREM OF PHYLLOTAXIS

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Fields of interest: Phyllotaxis, cognitive graphics, periodicity in chemical and nuclear physics, theory of music.
Publications:
Harmony of phyllotaxis, In: *International Conference "Mathematics and Arts" proceedings*, Moscow , 1997, pp. 218-222 (in Russian)
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Abstract: *In the concept of the Fundamental Theorem of the Phyllotaxis integrated and extended formulas are present for Fibonacci type series beginning with any integer U_1. The formulas have empirical nature. Application of these formulas results on the certain step in dense packing with additive properties typical for phyllotaxis.*

1 FUNDAMENTAL THEOREM OF THE PHYLLOTAXIS

The Fundamental Theorem of the Phyllotaxis (called so by Roger V. Jean) is understood as a number of the statements concerning interrelation of a divergence angle with parastichy number on families of a spiral phyllotaxis pattern. Key items of these statements are formulas that describe normal (for additive Fibonacci type series, starting with $U_1=1$) and anomalous (for the additive series starting only with $U_1=2$) phyllotaxis. Cases when $U_1=>3$ were been named aberrant and problematic patterns.

1.1 Glossary

Phyllotaxis: The arrangement of leaf or floret primordia at the shoot apex, or on the stem. In 92% of the observation the contact on conspicuous parastichy pair (m, n) in phyllotactic patterns is such that m and n are consecutive terms of the Fibonacci sequence, and the divergence angle d converges rapidly toward the Fibonacci angle. When m and n do not belong to the Fibonacci sequence, they are generally consecutive terms of a Fibonacci-type sequence. This constitutes a part of the challenge of phyllotaxis.

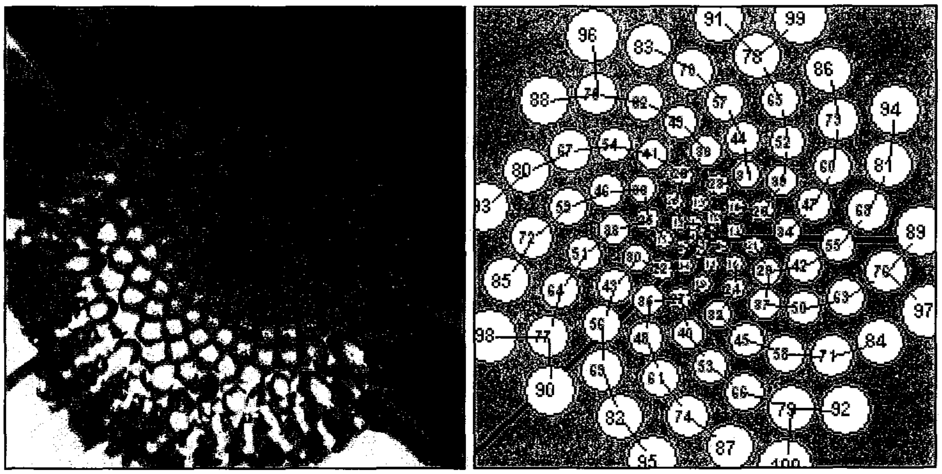


Figure 1: Daisy florets are arranged in a pattern of sets of parastichies, which number …8, 13, 21 in the outer part. Along each spiral from a family of n spirals differ by n.

$F_1 = 1$; $F_2 = 1$; $\langle 1, 1, 2, 3, 5, 8, 13, 21\ldots\rangle$; $D = 222.5°$; $d = -137.5°$. $|D| + |d| = 360°$

In new terms: $s = 1$; $t = 2$; $U_1 = s$; $U_2 = st - 1$.

Fibonacci sequence: The sequence of integers: 1, 1, 2, 3, 5, 8, 13, … The *k*th term is built using the rule $F_k = F_{k-2} + F_{k-2}$ from the initial values $F_1=1$; $F_2=1$.

Fibonacci-type sequence: A sequence built from the same recurrence rule as the Fibonacci sequence but starting with other initial terms. For example, Lucas sequence:

1, 3, 4, 7, 11, 18, 29, 47, …; $U_k = U_{k-2} + U_{k-1}$; U_1 , U_2 – integers ; n = 1, 2, 3, …

Golden ratio: $\alpha=(1+\sqrt{5})/2=1.6180339\ldots$; $\beta=(1-\sqrt{5})/2 = -0.6180339\ldots$; $\tau = \alpha$; $\tau^{-1} = -\beta$.

Divergence angle d: Any of the two angles at the center of a transverse section of a growing shoot tip determined by consecutively initiated primordia. (This definition differs from those, given by Roger V. Jean, a little).

Fibonacci angle: The divergence angle $d = 360*\tau^{-1} \approx 137.5°$ or $D=360*(1-\tau^{-1}) \approx 222.5°$
Parastichy: Any phyllotactic spiral seen on plants, or in transverse sections of plant apices. If along a parastichy the numbers differ by n then we say that it is a n-parastichy.
Family of parastichies: Given any parastichy in the centric representation, the corresponding family is the set of parastichies with the same pitch, winding around a common pole in the same direction and going through the centers of all the primordia.

2 RESULTS - PREDECESSORS

Normal phyllotaxis	Anomalous phyllotaxis
first members U_1 and U_2 of sequence	
$U_1 = 1$	$U_1 = 2$
$U_2 = t$	$U_2 = 2t + 1$
divergence angles d	
$d = F_k/(F_k t + F_{k-1})$	$d = (U_k t + U_{k-1})/(2U_k t + U_{k+1} + U_{k-1})$
limit-divergence angles d	
$d = 360 (t + \tau^{-1})^{-1}$	$d = 360 (2 + (t + \tau^{-1})^{-1})^{-1}$

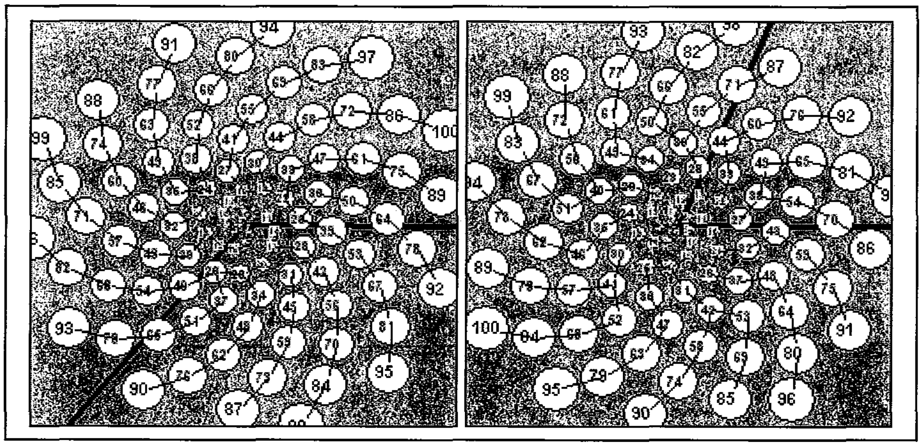
where $t \geq 2$ is an integer By R. V. Jean (1994, pp. 37-38)

3 NEW RESULTS

Variant 1	Variant 2
$U_1 = s$	
$U_2 = st + 1$	$U_2 = st - 1$
divergence angles	
$d = (F_{n-1} - U_n)/(U_1 U_n) =$ $= (F_{n-2} + F_{n-1} t)/(s (F_{n-2} + F_{n-1} t)+F_{n-1})$	$d = (F_{n-1} + U_n)/(U_1 U_n) =$ $= (F_{n-2} + F_{n-1} t)/(s (F_{n-2} + F_{n-1} t) - F_{n-1})$
limit-divergence angles	
$d = (1 - (U_2 - \beta U_1)) / (U_1(U_2 - \beta U_1)) =$ $= -(t-\beta) / (s (t-\beta)+1)$	$d = (1 + (U_2 - \beta U_1)) / (U_1(U_2 - \beta U_1)) =$ $= (t-\beta) / (s (t-\beta)-1)$

where s and t are integers

Notice that on figures in central area the circles are not evenly distributed, but in the outer part a packing become denser. By analogy, a ratio of two consecutive Fibonacci-type numbers U_k and U_{k+1} approaches Golden ratio as k become large.



a b

Figure 2: (a) $s = 3$; $t= 4$; $U_1 = s = 3$; $U_2 = st - 1 = 11$; $U_3 = 14$; $d=-129.34°$; $D=230.66°$
(b) $s = 5$; $t= 2$; $U_1 = s = 5$; $U_2 = st + 1 = 11$; $U_3 = 16$; $d = 66.89°$; $D=-293.11°$

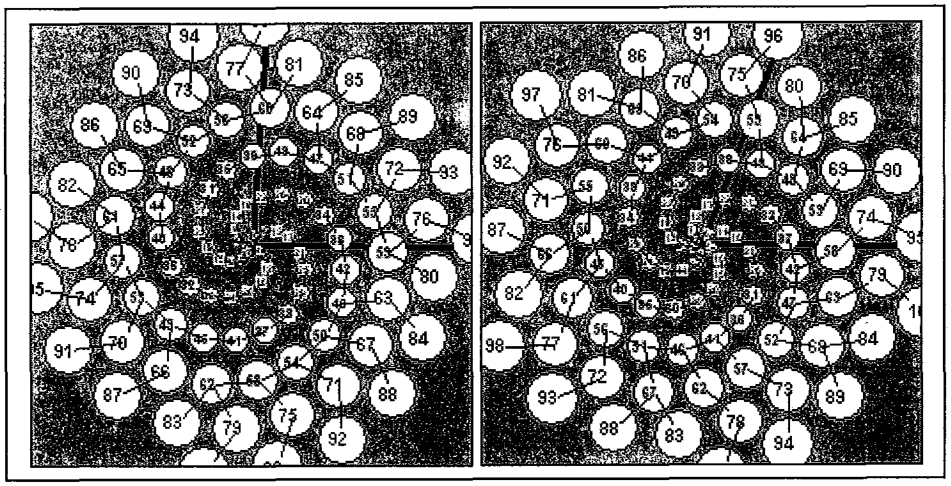


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Figure 3: (a) $s = 4$; $t= 4$; $U_1 = s = 4$; $U_2 = st + 1 = 17$; $U_3 = 21$; $d = 85.38°$; $D=-274.62°$
(b) $s = 5$; $t= 3$; $U_1 = s = 5$; $U_2 = st + 1 = 16$; $U_3 = 21$; $d = 68.23°$; $D=-291.77°$

References

Roger, V. Jean. *Phyllotaxis: A systemic study in plant morphogenesis,* Cambridge University Press 1994.