

SELF-REVERSED CONFIGURATIONS ON CUBE

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Abstract: *The number of patterns for painting six faces, eight vertices, and 12 edges of a cube with two colors black and white is shown respectively. At first, we considered symmetry as only the rotation of the cube, but then added the reflection in the cube, and finally the interchange of the two colors. Such interchange distinguishes only contrast of colors. If the interchange maintains the same contrast, we call it "self-reversed configuration". We counted the number of self-reversed configurations in cube coloring, and found that there are two self-reversed configurations in face coloring, six self-reversed configurations in vertex coloring, and eight self-reversed configurations in edge coloring.*

1 ENUMERATION IN COMBINATORIAL MATHEMATICS

Enumeration in combinatoril mathematics is important in modern science and technology. For example, it was applied in physics (Mayer and Mayer 1940), chemistry (Pólya 1937), information science (Harrison 1965), computer graphics (Lorensen and Cline 1987), etc. .

Recently, the authors investigated the enumeration of cutting point configurations in cube cutting (Teshima et al. 2004), which provided the mathematical foundation of Volume CAD (VCAD) as a next generation computer-aided design (CAD) tool (Kase et al. 2003). The enumeration process involved was equivalent to finding the distinct ways of painting the twelve edges of a cube with two colors.

In this short paper, enumerations for painting six faces, eight vertices, and 12 edges of a cube with two colors are described in Section 1.1, and “self-reversed configuration” is discussed in Section 2.

1.1 Painting cube with two colors

We summarized how many equivalent classes (= patterns) there are when we paint six faces, eight vertices, and 12 edges of a cube using two colors black and white. If the symmetry of the cube were not taken into account, there were $2^6=64$ different configurations altogether for face coloring, $2^8=256$ different configurations altogether for vertex coloring, and $2^{12}=4096$ different configurations altogether for edge coloring. We considered both symmetric group the rotation group and reflection group of a cube. We calculated the number of patterns in all configurations using the Pólya's theory of counting (Redfield 1927 and Pólya 1937), which is useful for enumeration under group action. In most applications of the Pólya theory, only the rotation group is considered and then enantiomorphous pairs which are mirror images of each other are counted individually. However, from the standpoint of pure geometry or pure group theory, such distinction is superfluous and it is proper to add the reflection group for reducing them (Coxeter 1989).

Enumeration results are shown in Table 1, 2, and 3. The second row in the tables shows the number of patterns under the rotation group only, and the third row shows the number of patterns under both the rotation and reflection groups. On face coloring, all 64 configurations are classified into 10 patterns, all 256 configurations are classified into 23 or 22 patterns on vertices coloring, and all 4096 configurations are classified into 218 or 144 patterns on edge coloring.

Table 1: Number of patterns for painting six faces of cube using black and white
Enumeration results under both rotation and reflection groups are the same as the number under the rotation group because enantiomorphous pairs do not appear in face coloring.

Number of black faces	0	1	2	3	4	5	6	Total
Number of patterns (rotation only)	1	1	2	2	2	1	1	10
Number of patterns (rotation & reflection)	1	1	2	2	2	1	1	10

Table 2: Number of patterns for painting eight vertices of cube using black and white

Number of black vertices	0	1	2	3	4	5	6	7	8	Total
Number of patterns (rotation only)	1	1	3	3	7	3	3	1	1	23
Number of patterns (rotation & reflection)	1	1	3	3	6	3	3	1	1	22

Table 3: Number of patterns for painting 12 edges of cube using black and white

Number of black edges	0	1	2	3	4	5	6	7	8	9	10	11	12	Total
Number of patterns (rotation only)	1	1	5	13	27	38	48	38	27	13	5	1	1	218
Number of patterns (rotation & reflection)	1	1	4	9	18	24	30	24	18	9	4	1	1	144

2 SELF-REVERSED CONFIGURATION

When interested only in the contrast patterns in Table 1, 2, and 3, we further reduce the total number of patterns. In face coloring, for example, in the case of (number of black faces, number of white faces) = (0, 6), (1, 5), and (2, 4), their contrast patterns are the same as the cases: (6, 0), (5, 1), and (4, 2). Thus, the former patterns can be reversed into the latter patterns. Here, "reverse" means the interchange of the two colors black and white. We need careful judgment for two patterns in the case of (3, 3) in Table 1. The same situation occurs in vertex coloring and edge coloring. We must judge six patterns in the case of (number of black vertices, number of white vertices) = (4, 4) in Table 2, and 30 patterns in the case of (number of black edges, number of white edges) = (6, 6) in Table 3.

As a result, each of the two patterns of (3, 3) in face coloring is self-reversed configuration, and each of the six patterns of (4, 4) in vertex coloring is also self-reversed configuration. Concerning the 30 patterns of (6, 6) in edge coloring, only eight patterns are self-reversed configurations and the other 11 patterns are reversed into the remaining 11 patterns. Figure 1, 2, and 3 show these results. The index with an underline shows that the figure is self-reversed. (And the index with an asterisk in the figures shows that the figure is one of the enantiomorphous pairs.)

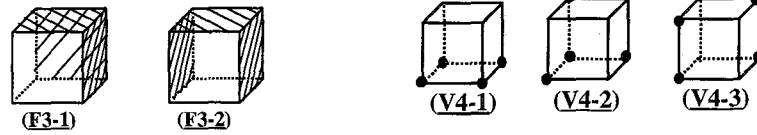


Figure 1(above): Two contrast patterns of (3,3) in face coloring

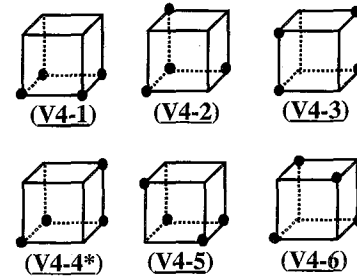


Figure 2(right): Six contrast patterns of (4,4) in vertex coloring

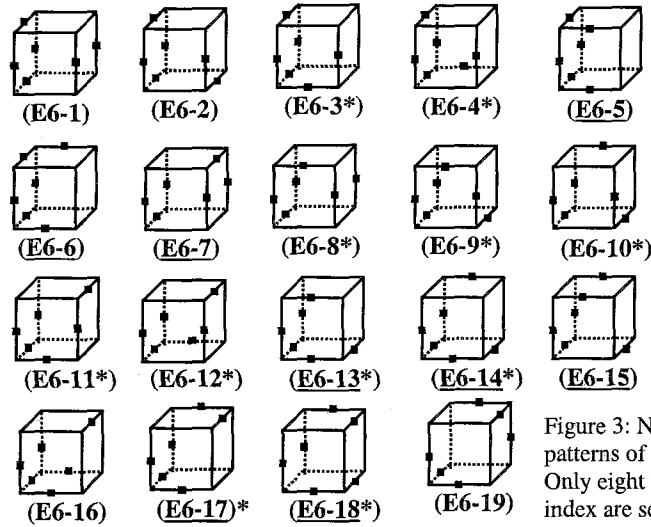


Figure 3: Nineteen contrast patterns of (6,6) in edge coloring. Only eight figures with underlined index are self-reversed.

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