Symmetry: Art and Science
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OPTIMAL GEOMETRY AS ART

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Exhibit *Art and Mathematics 2000* at the Cooper Union in Manhattan, NY, USA, Nov-Dec 2000.
Exhibit *Matematica, Arte e Cultura* at the Universita di Bologna, Italy, Oct-Dec 2000.
Exhibit *Intersculpt 2001*, Dayton, OH, USA, Nov 2001-Jan 2002.

***Abstract**: This essay is an updated version of one originally written for Math Awareness Month, April 2003, when it appeared online. It considers various relations between art and mathematics, especially the mathematics of optimization problems in geometry, and investigates certain issues arising in mathematical visualization.*

1 WHAT IS MATHEMATICS?

Mathematics can be defined as the study of abstract patterns. Numbers, of course, are one of the first examples of an abstraction: two plus three is five, whether we're adding apples, or oranges, or something abstract like ideas. But for mathematicians, patterns of even greater abstraction are found all around us.

Symmetries form another familiar example of patterns. The human visual system is especially sensitive to vertical mirror symmetry, as found (approximately) in the faces and bodies of most animals.

1.1 Mathematics and Art

Mathematics can be related to art in many ways. One can study art mathematically, looking for symmetries or other relations in the construction of a painting or sculpture.

Conversely, mathematical algorithms can be used to help create art: fractal systems, for instance, can recreate realistic shapes of plants, mountains and clouds.

Famously, perspective drawing has a mathematical basis, and is a good example of how different the human brain is from a digital computer. It is trivial for a computer to apply the rules of perspective to project a three-dimensional model world to a two-dimensional drawing, while human artists often have difficulty applying these rules. On the other hand, we effortlessly use our visual system to reconstruct a three-dimensional model of the world around us from the two-dimensional images presented on our retinas. We thereby solve a very difficult (even ill-posed) problem that the best computers still have trouble with.

1.2 Mathematics in the Natural Sciences

Perhaps the Pythagoreans were the first to suggest that "at its deepest level, reality is mathematical in nature". Although by today's standards their mathematical tools were limited, they found interesting numerical patterns in musical harmonies as well as in geometric figures.

The flourishing of science in Europe during the Renaissance was made possible by increasing knowledge of mathematics. Galileo echoed Pythagoras when he observed that the laws of nature are "written in the language of mathematics".

It is surprising how often branches of pure mathematics that seem to have no application to the real world turn out later to be very important in physics or other fields. Eugene Wigner described this as "the unreasonable effectiveness of mathematics in the natural sciences", in his famous 1960 essay.

1.3 Optimization Problems

Perhaps one reason for this effectiveness of mathematics is that many laws of physics can be expressed in terms of minimizing free energy or minimizing action. These optimization problems have mathematical solutions.

In general, surface energies become more important than bulk energies at small scales: A small bug can easily walk on water, because the force of surface tension outweighs gravity at that small scale.

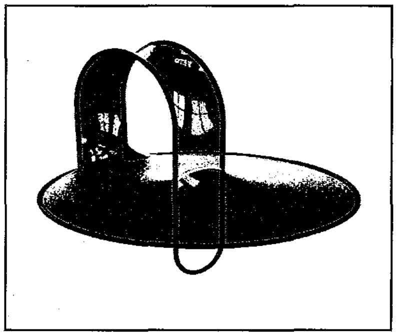
Thus problems about real-world materials, especially those concerning structure at small scales, can often be cast in the form of optimizing some feature of shape. The system minimizes some energy depending on the shape of a surface (or sometimes a curve) describing the material's structure. Mathematically, we obtain an optimization problem for some geometric energy.

A classical example is the soap bubble which minimizes its area while enclosing a fixed volume; this leads to the study of the more general constant-mean-curvature surfaces found in bubble-clusters and foams. Biological cell membranes, on the other hand, are

more complicated bilayer surfaces, and seem to minimize an elastic bending energy known as the Willmore energy.
My own mathematical research concerns geometric optimization problems like these. I have looked also at singularities in higher-dimensional soap films, configurations of points on a sphere, and ropelength of knots. Ropelength describes how to optimally tie a given type of knot in a piece of real rope, and it seems to be related to the physical behavior of knotted loops of bacterial DNA.

2 ΟΠΤΙΜΑΛ ΓΕΟΜΕΤΡΨ ΑΣ ΑΡΤ

Two thousand years ago, Seneca wrote that "All art is but imitation of nature". Assuming this, if nature can be described mathematically, the same should be true of art. Of course, current-day mathematics is not sufficient to describe truly complex systems, like humans and their relationships, which are the subjects of much great art. And perhaps some of these areas will forever stay outside the realm of mathematical analysis. Often abstract art, including music, is closer to mathematics. Those aspects of nature, like optimization problems, that are most easily described in mathematical terms often lead to an abstract beauty. Henry James wrote that "in art economy is always beauty".
Like most research mathematicians, I find abstract beauty in the elegant and economical structure of mathematical proofs, and I feel that this elegance is discovered, not invented, by humans. I am fortunate, however, that my own work in optimal geometry leads directly to visually appealing shapes, which can present a kind of beauty more accessible to the general public.



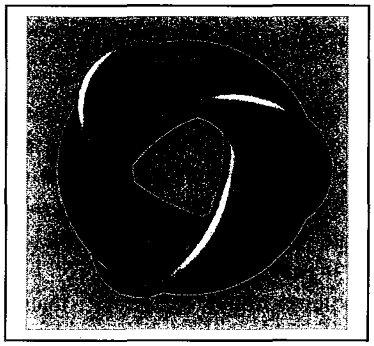


Figure 1: This soap film (left) spanning a bent wire boundary, has been rendered with a custom soap-film shader for Renderman. This trefoil or overhand knot (right) sits on the surface of a torus, illustrating why mathematicians also call it a (2,3)-torus knot. Its shape has been optimized by letting the different strands repel each other with a Coulomb-like repulsive force.

Figure 2: This minimax sphere eversion from *The Optiverse* is a geometrically optimal way to turn a sphere inside out, minimizing the elastic bending energy needed in the middle of the eversion.

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