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# HYPERBOLIC TESSELLATIONS BY *TESS*

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***Abstract:*** *This paper provides an overview of methods for creating symmetric patterns based on tessellations and applications of isometries to basic objects of hyperbolic plane. Algorithm for creating and drawing different types of planar edge-to-edge tessellations of Euclidean, elliptic and hyperbolic plane with regular polygons as tiles (not all congruent) and vertices of the same type, including regular, uniform, non-uniform and coloured tessellations is implemented in "Mathematica 4.0"® package "Tess". Using tools provided in the package we focus on discovering the intrinsic abstract beauty of mathematics by creating more intricate tessellations. Automatically generated coloured tessellations of the hyperbolic plane presented in a Poincare disk model, together with additional data provided, are combined with an arbitrary additional motif in order to obtain a new design. Based upon a choice of the subpattern/motif the same tessellation will become more/less symmetrical and can appear totally different.*

## 1 HYPERBOLIC POLYGONS, ISOMETRIES, TESSELLATIONS

Pattern designs created in different cultures have been a subject of mathematical analysis and only recently they have been created (using computers) based on the new mathematical knowledge like hyperbolic geometry. Visualization of hyperbolic plane, even of the basic objects and their images under isometries, can be highly symmetrical and visually attractive (Fig. 1).

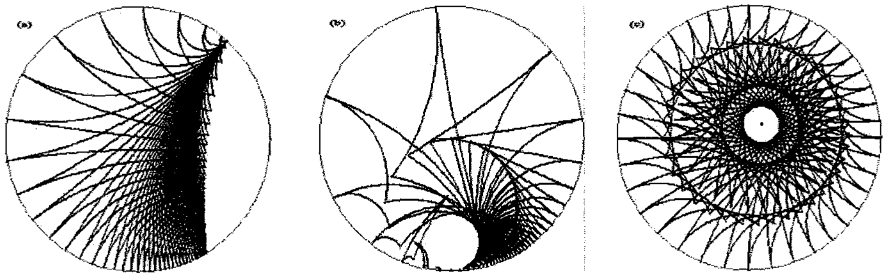

Figure 1. Translation of (a) a triangle and (b) square, and (c) rotation of a triangle around the finite point in Poincare disk model.

Using *Mathematica* package *Tess* [6] different types of tessellations and all possible realizations can be generated: uniform (Fig. 2a, 2b), chiral uniform (Fig. 2c) and chiral non-uniform (Fig. 2d). There are no limits on the choice of polygons or the number of polygons around each vertex. Moreover, polygons of the same colour can be mapped on each other by isometries which do not change the tessellation, or we can obtain various "coloured" tessellations.

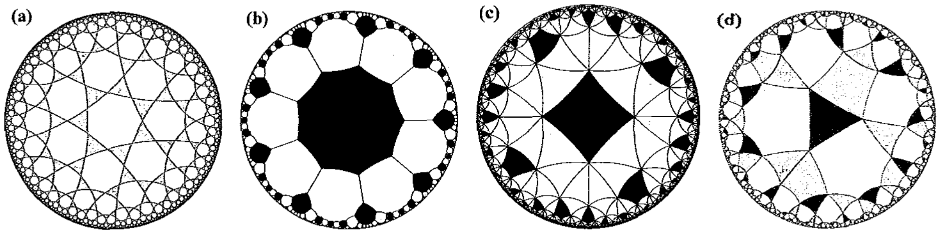


Figure 2. Uniform hyperbolic tessellations: (a) (3,7,3,7); (b) (9,12,12) ; (c) chiral (3,3,3,3,3,4); (d) non-uniform chiral (3,4,6,3,6).

## 2 TESSELLATIONS AND PATTERNS

Images obtained by tessellating hyperbolic plane usually show simple symmetric patterns and now we will focus on creating more intricate ones. As a base we use tessellation given with its vertex type $(n_1, n_2, \ldots, n_N)$ meaning that the polygons about this vertex in cyclic order are an $n_1$-gon, an $n_2$-gon, . . . and an $n_N$-gon, where $N$ is the number of polygons at each vertex. By connecting the incenters of adjacent polygons/tiles surrounding a vertex, an $N$-gon is formed. Basic domain is this $N$-gon or one of the polygons of its decomposition into congruent polygons based on its internal symmetries. In the most cases it is the actual fundamental domain (or several adjacent fundamental domains connected), but it is always sufficient to construct the whole tessellation. The transformation rules of the tessellation combined with an arbitrarily chosen motif contained in the fundamental domain (Fig. 3c) enables us to create infinite number of variations of every hyperbolic tessellation.

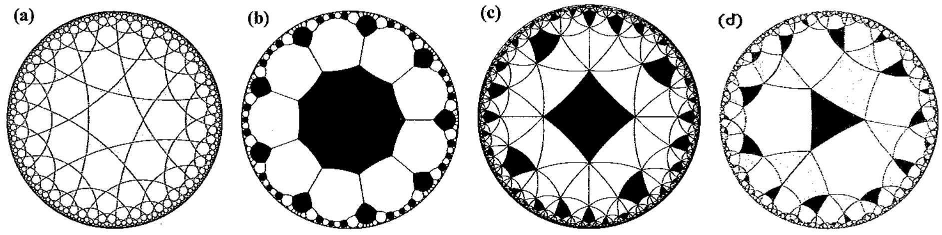


Figure 3. (a) Uniform tessellation (4,4,4,6); (b) basic domain with a motif superimposed with the original tessellation; (c) the motif contained in basic region.

Based on a choice of the motif, a single tessellation can appear totally different (Fig. 4): manifest symmetries that were not present, stay the same if the motif exhibits local symmetries consistent with the symmetries of the entire tessellation while asymmetric motif can hide symmetries that tessellation admits.

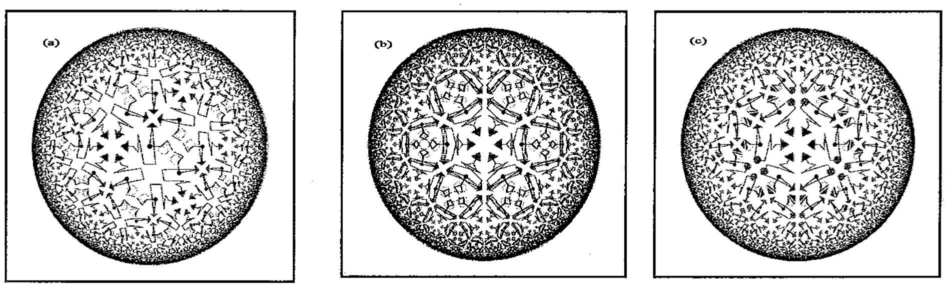

Figure 4. Tessellations obtained from (4,4,4,6) using different patterns.

The same method can be applied to hyperbolic tessellations presented in Poincare half-plane and Klein disk models, or to Euclidean or spherical tessellations. However, this is still relatively unexplored area with many open questions: finding all different possible shapes of the fundamental region, analysing knot patterns and aperiodic tessellations, *etc*. Once the whole method is implemented, there will be more room for personal artistic expression, in order to emphasize intrinsic beauty of hyperbolic geometry and its interrelations with art and design.

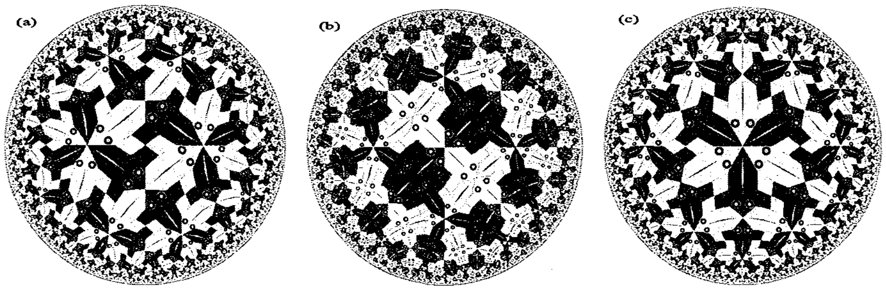


Figure 5. Escher-like tessellations based upon uniform tessellations: (a) (6,6,6,6,6,6); (b) (6,6,6,6); (c) (4,6,4,6).

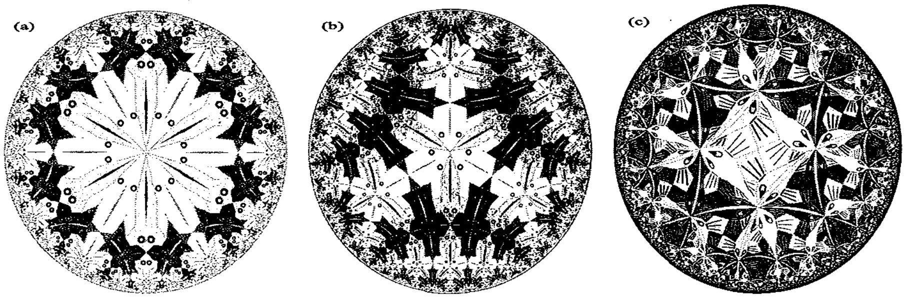


Figure 6. Escher-like tessellations based upon uniform tessellations: (a) (4,4,6,4,4,6); (b) (4,4,6,6); (c) (3,4,3,4,3,4).

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