

SPACE CURVES WITH CONSTANT THICKNESS: THE CASE OF CIRCULAR HELIX

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Abstract: Aiming a new type of educational attempt to inspire some developing ideas, the author enjoys a kind of designs by the tool of 3D graphics of Mathematica and Maple. He starts the process with giving constant thickness to simple spatial curves. He himself does calculate and draw graphics but these are not the main subject at all for students, who should enjoy the interesting properties, raising some questions and ideas for the next step. He thinks both of the calculation and graphics work as the next task after the students feel necessity for that. The main part of the presentation is to show the interesting properties of the 3D-curves with constant thickness. First of all, we start with the 2D-correspondence of the problem. The concept of helical symmetry is defined that is proper to the present problem.

1 INTRODUCTION

Nowadays, too many people in Japan regard studying mathematics and physics as for rote learning of formula. Most of scientists and mathematicians take the situation as serious. We should construct more creative education. It must be changed in order to be more creative one. This work is one of the attempts to this direction and at the same time, he enjoys by himself to study and to images the detailed structure of ideal 3D structures.

2 THICKNESS OF A LINE IN 2D and 3D

A line segment is a straight line of finite length l . It is reasonable to define a line segment with 2-dimensional thickness or width w as a rectangle and the area of a rectangle is width by length wl . Similarly, it is reasonable to define a circumference of radius r with width w as a pair of concentric circles whose radii are $r + w/2$ and $r - w/2$ since the area is $\pi(r + w/2)^2 - \pi(r - w/2)^2 = w \times 2\pi r$ as expected.

It is the most reasonable extension to define the corresponding width in 3D as the circle in the perpendicular plane with its centre at the point on the curve. The definition is acceptable at least for the torus $(\sqrt{x^2 + y^2} - a) + z^2 = b^2$ that is formed by rotating the smaller circle of radius b with bigger radius a , whose volume is given by $V = 2\pi^2 ab^2 = (\pi b^2) \times (2\pi a) = \text{width} \times \text{length}$. The easiest and the most elegant way to understand the relation is that the sections for any constant value of z where $|z| < b - a$ is a pair of concentric circles by reducing to 2D case.

3 THE SIMPLEST SPATIAL CURVE: CIRCULAR HELIXES

A straight line is the most homogeneous line with a constant direction but without any curvature and can be in one dimension. A circle is the most homogeneous line with a constant curvature but without any twist (rotation + translation) and is two-dimensional. A circular helix is the simplest essentially three-dimensional curve, being the most homogeneous line with a constant twist, never realize in lower dimensions than three. The curve in Cartesian coordinate is given by $(a \cos(kz), a \sin(kz), z)$ and in cylindrical coordinate (r, φ, z) by $r(z) = a$, $\varphi(z) = kz$ where $a(>0)$ stands for the radius of the cylinder on the surface of which the circular helix lies and constant k depends on the pitch. Regard z as time, then it is nothing but pure rotation in a plane. The torsion of the curve (Coxeter, 1961, 1969, pp. 323-325, where $c \equiv 1/k$ is used) is given by $k/(a^2 k^2 + 1)$. A helical surface can be regarded as a set of the circular helix with common value of k and for arbitrary value of $a(>0)$. It is defined as the locus of rotating horizontal straight line around a fixed vertical line. A circular

helix is the section of a helical surface and a cylinder of radius a . It is natural to introduce the concept of *helical symmetry* for the circular helixes belonging to a common helical surface. Reminisce that a step stone of helical staircase is triangular (neglecting the limiting part close to the central axis). The slope of the staircase depends on the distance from the axis; infinite at the axis and tends to zero in infinite remote.

Let us focus on the problem of giving a single circular helix a thickness. Now, a is a constant and an angle $\alpha \equiv \tan^{-1}[1/(ka)]$ is introduced. Prepare a sheet of paper on which parallel lines of slope angle α are drawn with the periodic separation of $2\pi a \sin \alpha$ (the corresponding period in intercept is $2\pi/k = 2\pi a \tan \alpha$) as shown in Fig.1. Then, put it on the surface of cylinder so that the horizontal axis just fits around the cylinder of radius a and then the parallel lines make a single slope. For $\alpha = 0$ and then $c = 0$, a circle is obtained. For $\alpha \rightarrow \pi/2$ then $c \rightarrow \infty$, the helix tend to a straight line corresponding to the generating line of the cylinder.

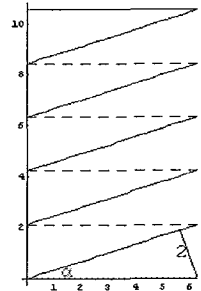


Fig. 1

3.1 The densest helixes with thickness

Though the value of α is arbitrary as a line, there exists the minimum value if the finite thickness is given to the helix and then the volume exclusion effect is introduced. Obtaining the minimum value of slope angle is rather complicated. All the circular sections are inclined and there are no diameters parallel to the axis line. It means that the ends of diameters tend to go remote but radial ones. This effect in 3D space causes some confusion. Can you imagine the inner structures and the several planer sections, for example, horizontal and vertical? Another case that can be treated in parallel is double helix. In the cases of more than two helical lines, there are vacant space around the central axis, as easily seen from the fact that there are no point in a plane that three impenetrable circles can share. Therefore it is necessary to start the analysis with how three cylinders symmetrically contact one another.

3.2 How are the fine structures inside?

It is interesting to imagine the hidden fine structures inside. For example, how a circular section contact with another circular sections and how they arrange? Interesting problems arise step by step. From an educational point of view, it is desirable to have, to imagine the situation, and to enjoy such kind of questions even if they cannot find the

answer. It will motivate towards many directions such as mathematics, science, designing etc.

The procedure for the author to answer those problems is based on the fact that a helices with thickness is set of helices with common value of α . All of the helical lines with α that passes in inside of a circle go keeping the whole shape. Then one can trace all of the positions on the trajectories or flow line. See the figures here showing the situation. Some of the figures here are exhibited with some explaining text at gallery. More figures containing some sections will be added in the oral presentation.

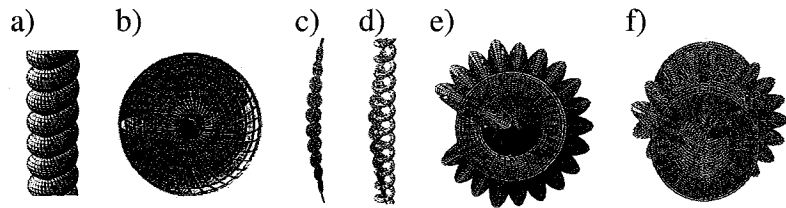


Figure 2 The densest single helix (left to right) a) from a side, b) from upper, c) a part of touching discs, d) the same discs with the core helix, e) the same as just before from upper, f) the same from a slightly shifted direction.

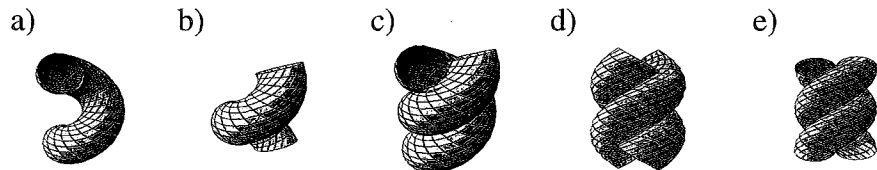


Figure 3: The densest double helix (left to right) a) one of two elements, b) the other element, c) combining them, d) view of the combined one from a symmetrical direction, e) the same from the opposite side.

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