Symmetry: Art and Science
Tihany Congress, 2004

A POLYHEDRON WITHOUT DIAGONALS (1949)

ÁKOS CSÁSZÁR

Name: Ákos Császár, Mathematician (b. Budapest, Hungary, 1924).
Address: Honorary President, Bolyai János Mathematical Society, Budapest, Hungary; Professor Emeritus, Eötvös Loránd University, Budapest, Hungary; *Home address*: Budapest, Párizsi u. 6/A,. H-1052 Hungary.
Fields of interest: Topology (Hobby: Music, botany, bridge card-game).
Awards: Kossuth Prize, 1963 [a highly regarded Hungarian state prize given for creative work in all fields of art, science, technology, agriculture]; many further awards and medals for scientific work.
Publications: Monographs on general topology in French, 1960, in Hungarian, 1970, and in English, 1978; many textbooks; more than 100 scholarly papers.

***Császár Polyhedron [Note by the Editor]**: During the opening ceremony of the congress, we celebrate the 80th birthday of Ákos Császár and 55th anniversary of the discovery of the Császár polyhedron. The latter topic was inspired by a problem in a mathematical competition for students in 1948: Prove that there is no polyhedron, beyond the tetrahedron, where any two vertices are connected by an edge. In other words, the tetrahedron is the only polyhedron without diagonals. Indeed, this statement is valid for convex polyhedra. However, Császár gave a concave example that has no diagonal. Interestingly, his paper of 1949 had no figure, which is not surprising if we consider the difficulties at Hungarian universities after World War II. Perhaps, the lack of figures contributed to the fact that this paper did not inspire many further studies until Donald W. Crowe's contributions in the early 1970s and then Martin Gardner's popular paper on the "Császár polyhedron" in the* Scientific American *(May 1975). Later L. Szilassi and more recently J. Bokowski significantly contributed to this topic; see their papers in this volume. We celebrate the double-jubilee by the first ever Császár-Crowe-Szilassi-Bokowski meeting and by reprinting Császár's paper of 1949. Note that this is not in the main stream of Császar's works in general topology, but still his favorite among the "other" contributions (cf., the interview and papers in the "Supplement" to the journal* Természet Világa, *November 1994, in Hungarian, which celebrated the 70th birthday of Császár). [D. Nagy]*

Separatum.
ACTA UNIVERSITATIS SZEGEDIENSIS
SECTIO SCIENTIARUM MATHEMATICARUM
REDIGUNT:
L. KALMÁR, B. SZ.-NAGY, GY. SZ.-NAGY,
L. RÉDEI ET F. RIESZ

ACTA
SCIENTIARUM
MATHEMATICARUM

TOMUS XII.
FASC. 2.

A polyhedron without diagonals.

By ÁKOS CSÁSZÁR in Budapest

It is simple to prove that *the tetrahedron is the only polyhedron homeomorphic to the sphere and having the property that every two of its vertices are joined by an edge.* In fact, such a polyhedron must be triangle faced; and if we denote by ν the number of its vertices, then it has $\binom{\nu}{2}$ edges and $\frac{2}{3}\binom{\nu}{2}$ faces, so that the theorem of EULER gives

$$\frac{2}{3}\binom{\nu}{2}+\nu-\binom{\nu}{2}=2. \qquad (1)$$

This equation furnishes $\nu=3$ or $\nu=4$; the first solution has no geometrical meaning and the second gives the tetrahedron.

The question arises, whether this proposition remains true or not by omitting the restriction concerning the topological type of the polyhedron. We shall give in this paper a negative answer to this question by showing the existence of a polyhedron homeomorphic to the torus with the property mentioned above.

For the case of the torus, we have to put 0 instead of 2 on the right side of (1) and we obtain from this equation $\nu=7$. We first shall draw the 7 vertices and the 21 edges of our polyhedron on the torus.

Let us represent the torus on a rectangle $ABCD$. The opposite points lying on the sides of this rectangle are the images of the same point of the torus. Let us take seven points $1, 2, 3, \ldots 7$ in this order on the side AB; they appear naturally on the opposite side CD too. By drawing the straight segments joining the point 1 (on AB) to the points 3 and 4 (on CD), then those joining the point 2 (on AB) to the points 4 and 5 (on CD) and so on in the cyclic order of the vertices, these segments together with the segments of AB (and CD) joining two neighbouring vertices, form a system of lines containing 21 [illegible] which joins every pair of the seven vertices by an edge [illegible] the torus represented by the rectangle $ABCD$ in 14 [illegible] enumerates the vertices of these 14 triangles [illegible]

Table 1.

126	2 5	356	346	467	237	267
156	2 5	124	134	137	457	157

We shall now construct a polyhedron which realizes this topological scheme.

Table 2 shows the coordinates of the seven vertices of our polyhedron in a rectangular system of coordinates. The values of a and b will be given later on. This system of vertices shows an axial symmetry with respect to the z-axis, 1 and 6, 2 and 5, 3 and 4 corresponding to each other. As table 1 shows, the faces of the polyhedron show the same symmetry. We have to choose the values a and b in the way that no pair of the faces intersect each other.

Table 2.

	x	y	z
1	-3	3	0
2	3	-3	a
3	1	2	3
4	1	2	3
5	3	3	a
6	3	-3	0
7	0	0	b

In we put for a moment $a=0$ and $b=+\infty$, a short computation shows that the plane passing through the vertices 356 divides the space in two half spaces in the way that the points 1 and 2 lie in the first half space and the points 4 and 7 in the second. We can use for the abbreviation of this fact the symbol

$$12 \mid 356 \mid 47 \quad \ldots \text{A.}$$

We get similarly

$$125 \mid 346 \mid 7 \quad \ldots \text{B}$$
$$145 \mid 236 \mid 7 \quad \ldots \text{C}$$
$$145 \mid 237 \mid 6 \quad \ldots \text{D}$$
$$23 \mid 167 \mid 45 \quad \ldots \text{E}$$
$$36 \mid 257 \mid 14 \quad \ldots \text{F.}$$

These propositions remain valid even if we give to a a sufficiently small and to b a sufficiently large p sitive value.

Denoting by $\overline{25}$ the plane passing through the points 2 and 5 and perpendicular to the z-axis, we have moreover

$$16 \mid \overline{25} \mid 347 \quad \ldots \text{G.}$$

We can now show that no two of the faces of our polyhedron intersect each other. Because of the symmetry with respect to the z-axis it suffices to consider pairs of faces formed by a face in the upper line of table 1 and an other which is written *before* it or *under* it in this table. Table 3 gives now for every pair of this type one of the propositions A — G which shows that these two faces cannot intersect each other.

Table 3.

126 —	156 E	345 —	245 B	237 —	126 C	267 —	156 C
235 —	126 G		356 B		156 C		235 C
	156 G		124 B		23 D		245 C
	245 F		134 B		245 D		356 C
356 —	126 A	467 —	126 B		356 C		124 C
	156 A		156 B		124 D		346 C
	235 A		235 B		346 C		134 C
	245 F		245 B		134 D		467 E
	124 F		356 B		467 E		137 D
346 —	126 B		124 B		137 D		237 D
	156 B		346 B		457 D		457 D
	235 B		134 B	267 —	126 C		157 D
			137 E				

Longer calculation shows that $a = 1$ and $b = 15$ satisfy our above conditions.

We mention finally that the generalized theorem of EULER shows the existence of an infinity of topological types for a polyhedron with the property that every two of its vertices are joined by an edge. It would be of some interest to investigate if all these types can be realized with polyhedra having plane faces and straight edges.

(Received February 10, 1949.)