

VISUALIZATION OF NUMERICAL PATTERNS BY MEANS OF CONGRUENCE RELATION

F. RUIZ AND L. RICO

Names: F. Ruiz and L. Rico

Address: Department of Didactics of Mathematic, University of Granada, Campus de Cartuja, 18071 Granada, Spain. *Phone:* (34)958243951 and (34)958243949. *Fax:* (34)958246359. *Email:* fcoruiz@ugr.es and lrco@ugr.es

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- Ruiz, F., Organizer of the exhibitions:

M. C. Escher, entre la Geometría y el Arte, University of Granada and Cordon Art, Granada. May 1990.

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Abstract: *The idea of Pattern occupies a central place nowadays in mathematics. Thus, many mathematicians agree on conceiving mathematics as "the science or the patterns" (Devlin, 1994). For Stewart (1995) we live in a Universe of patterns. The scientist's instinct is to try to understand the natural world, and that of the mathematician is to structure that process looking for the rule, the norm, the structure, that is to say, the pattern. The use of patterns in the teaching of the mathematics is very important, since patterns and regularities appear in the world that surrounds us as well as in the world of the mathematics.*

When congruence is considered in the Pascal Triangle, different triangular patterns appear, as a sort of fractal. An example is the Sierpinski triangle. We have studied the One Hundred Chart (natural numbers 1 to 100 in 10 rows and 10 columns) as a mean of visualizing additive operators with geometrical figures like poliminoes. Each class of those poliminoes is associated with an additive operator in the table. If we consider the congruence modulus m in the one hundred charts of k columns, rectilinear patterns

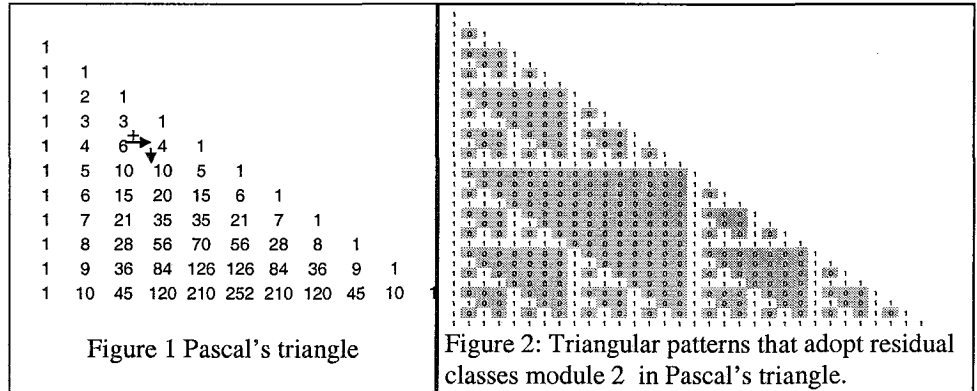
appear for each class of congruence. We vary both m and k and in this way we obtain a set of tables with regular patterns coloured with criterion of congruence. Now, the regularities founded visually in this table of tables can be expressed in a mathematical way. So that we obtain a translation between visual patterns and mathematical properties.

1 CONGRUENCE AND PATTERNS

The mathematical concept of congruence of numbers is a useful tool to detect and to identify numerical patterns.

The natural numbers can be classified according to the criterion “to obtain the same remainder when dividing them by m .” The numbers that produce the same remainder when dividing them by m , belong to the same class of equivalence or residual class, module m . In the particular case of dividing the numbers by 4, we obtain four residual classes module 4. The elements of the same class are characterized because their difference is always a multiple of 4.

Pascal’s Triangle is a triangular numerical table in which both lateral sides are filled by 1s and each element is the sum of the two numbers immediately above and to the left (Figure 1).



If in this table each number s is replaced by its residual class, we obtain triangular patterns.

In this paper we consider congruent numbers and residual classes on the one hundred chart or 100-Table

2 CONGRUENCE IN THE 100-TABLE AND THE RECTILINEAR PATTERNS

If we replace each number of the 100-Table with the remainder obtained when dividing it by 2, the numbers of this table are classified in two residual classes: the class of the even numbers and the class of the odd ones. The elements of these classes are aligned, although this alignment changes when the module m , as well as the number k of columns of the table, vary. The figures 4 shows three different rectilinear patterns formed in the 100-Table of 8 columns with one of the residual classes module 5.

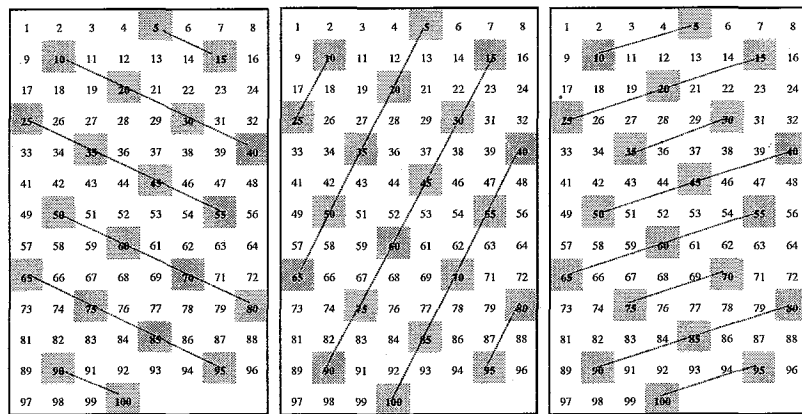


Figure 4: Different rectilinear patterns formed by one of the residual classes module 5 in the 100-Table of 8 columns.

In general, let us consider the 100-Table organized in k columns and let m be the module with which we will classify the numbers of the table. We call table (m, k) the classification obtained when considering the residual classes module m in this table of k columns, and we obtain m residual classes that we mark $\underline{1}, \underline{2}, \dots, \underline{m-1}, \underline{m}$.

If we use the same colour to highlight the numbers that are congruent, module m , we visualize the m resulting equivalence classes by means of different colours. Figure 5 (at the end of this paper) shows the tables (m, k) when m and k vary between 2 and 10. For their best observation we have highlighted only the elements of one class, since the other classes are placed “parallel” to it.

Our purpose is to study the regularities that are visualized in the tables (m, k) , from a visual-geometric point of view and with an algebraic focus using the concept of *chain*, emerged in an investigation of didactic character about the 100-Table (Ruiz, 2000).

3 PATHS AND CHAINS IN 100-TABLE

When some movements are carried out on the 100-Table, they can be interpreted as additive operations; displacing k cells to the right/left from a specific number is equal to adding/subtracting k units to/from that number.

Displacing k cells down/up from a number is equal to adding/subtracting k tens to/from such a number (figure 6).

1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100

Figure 6: Paths on the 100-Table

We call to those orientated paths, *chains* and we express them by the formula $C(c^+ d_{\pm})$, where c indicates the tens and d the units. The signs $+$ and $-$ provide the advance sense or setback of each stretch of the chain.

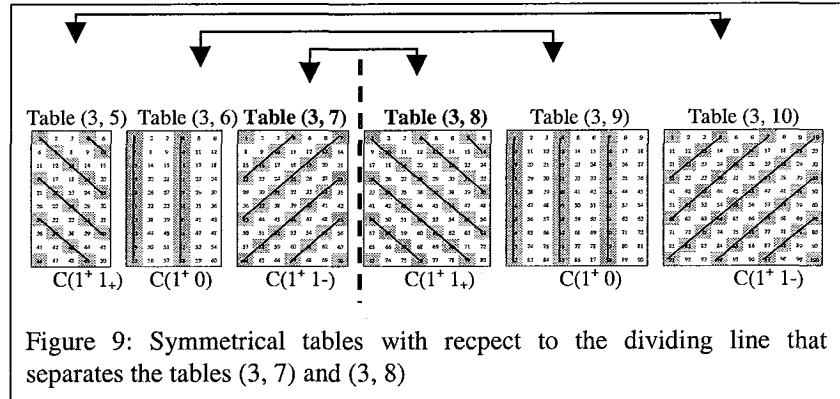
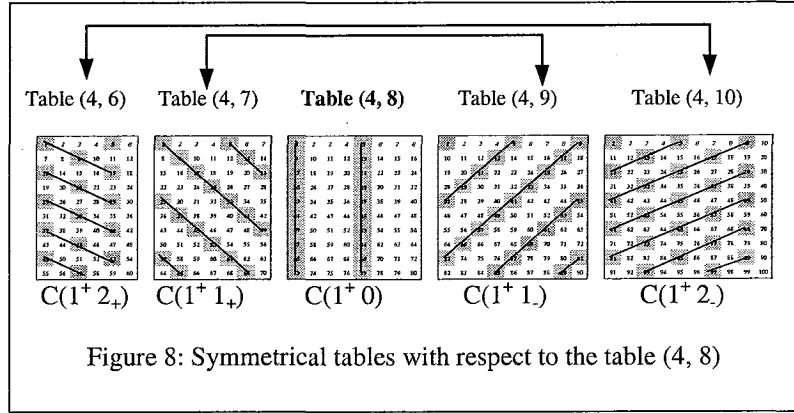
4 VISUAL PROPERTIES IN THE TABLES (m,k)

In order to express some of the regularities detected in the tables (m, k) (figure 5) we identify each rectilinear pattern with a chain $C(c^+ d_{\pm})$, named *generating chain* of the pattern.

We base the proves of the following properties that are visualized in the tables (m,k) (figure 5) on the mathematical expression corresponding to the previous sentence (see Ruiz; 2000).

1. If two tables have in common one pattern whose vertical component is a prime number $c > 1$ with the module m , then they have all the patterns of the table with the smallest number of columns in common. Such tables are called *tables with nested patterns*.
2. The tables corresponding to (m, m) that occupy the main diagonal in the figure 5, contain the pattern generated by the chain $C(1^+ 0_{\pm})$.

3. The tables where k is a multiple of m , (tables (m, m)) also have the pattern generated by the chain $C(1^+0_+)$.
4. The tables (m, k) and $(m, k+m)$ are tables with nested patterns.
5. If two tables (m, k) and (m, k') , are such that the number of columns of both k and k' are congruent module m , these tables are tables with nested patterns.



5 SYMMETRY PROPERTIES

Taking as a reference table (4, 8), we see that the previous and following tables have the patterns generated by $C(1^+1_+)$ and $C(1^+1_-)$ respectively, and they can be visualized as symmetrical patterns (figure 8).

We say that *two patterns are symmetrical* if their respective generating chains correspond to the notation $C(c^+d_+)$ and $C(c^+d_-)$.

We say that *two tables (m,k) and (m,k') are symmetrical* when, whatever the chain $C(c^+d_+)$ generating of a pattern in (m,k) , may be its symmetrical chain $C(c^+d_-)$ is a generating chain of a pattern in (m,k') , and reciprocally; with $d < \min(k, k')$.

S1. The tables that are equidistant of any table (m, m) , characterized to have the pattern generated by $C(1^+ 0)$, are symmetrical (figure 8).

S2. The tables that are equidistant of those that have the pattern generated by $C(2^+ 0)$ are also symmetrical.

S3. The tables equidistant from the pair of tables $(m, m\lambda+n)$ and $(m, m\lambda+n+1)$, with $\lambda = 0, 1, 2, 3, \dots$; $n = 1, 2, 3, \dots$, are symmetrical.

S.4 If a table (m, k) has the pattern generated by the chain $C(2^+ 0)$, then it is autosymmetrical.

In short, starting from the numerical field, we have changed to the visual-geometrical environment associating a chain to each additive operator. Once in the visual environment, we can state and prove the properties produced by the congruence in mathematical terms.

Table (4, 6)

1	2	3	4	5	6
7	8	9	10	11	12
13	14	15	16	17	18
19	20	21	22	23	24
25	26	27	28	29	30
31	32	33	34	35	36
37	38	39	40	41	42
43	44	45	46	47	48
49	50	51	52	53	54
55	56	57	58	59	60

$C(1^+ 2_+) \vee C(1^+ 2_-)$

Figure 10:
Autosymmetrical
table.

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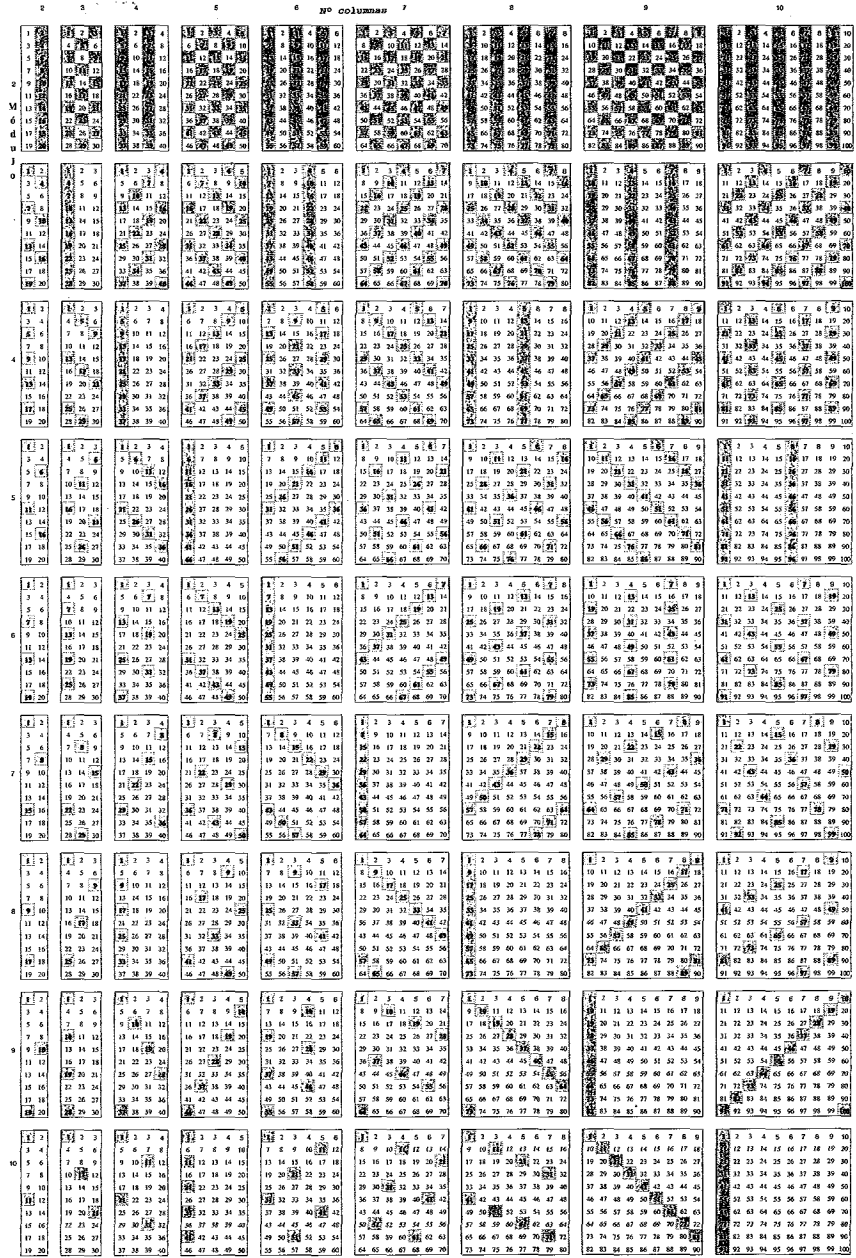
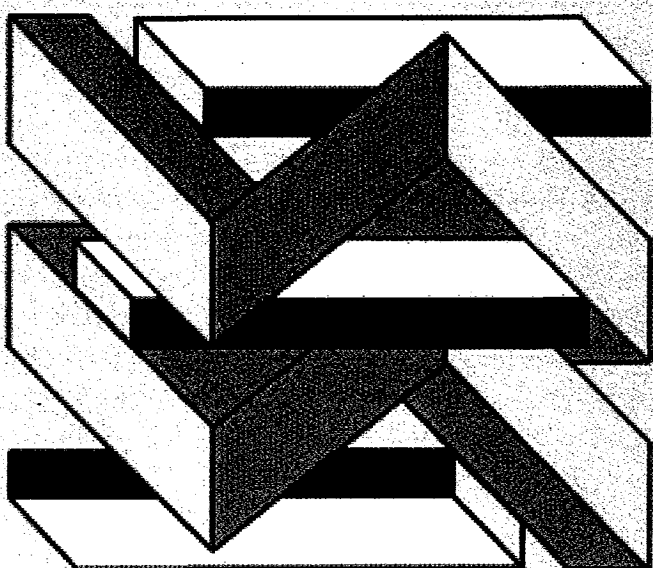


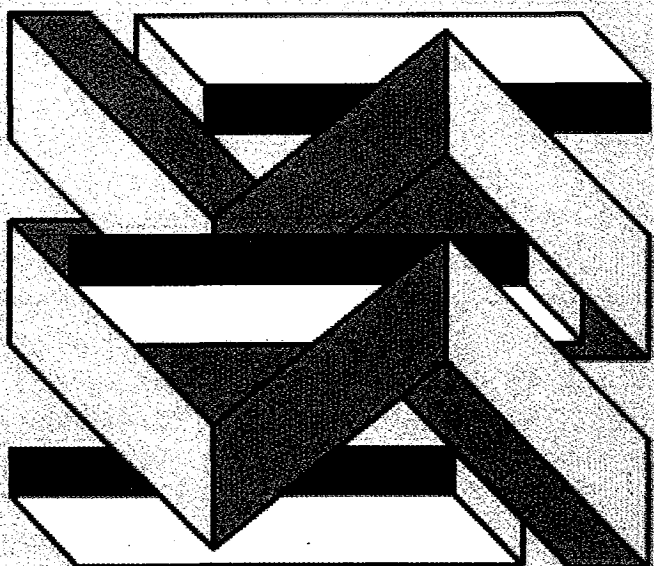
Figure 5: Rectilinear patterns in the tables (m, k)

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