

## **CURIE SYMMETRY PRINCIPLE IN NONLINEAR FUNCTIONAL ANALYSIS**

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**Abstract:** *The Curie symmetry principle states that the effect may occasionally have the same or a higher symmetry than the causes. But some nonlinear phenomena are found out to break this principle in observation. Especially, some of these phenomena dominated by nonlinear functional equations can be mathematically shown to break the Curie principle by Sattinger's group theoretical method. The result of the group theoretical method manifests both preserving and breaking the Curie principle occur in a nonlinear system.*

### **1 INTRODUCTION: THE CURIE PRINCIPLE**

The Curie symmetry principle (Curie, 1894) is the causality relation between the symmetry of the cause and that of the effect. The principle is composed of three parts:

- *If certain causes yield the known effects, the symmetry elements of the causes should be contained in the generated effects.*
- *If the known effects manifest certain dissymmetry (absence of symmetry elements), this latter should be contained in the causes which have generated those effects.*

- *The converse to these two previous propositions is not true, at least in practical: i.e., the effects may have higher symmetry than the causes which generate these effects.*

The Curie principle is mentioned and restated by many following researchers (Jaeger, 1917; Prigogine, 1947; Groot and Mazur, 1963; Chalmers, 1970; Shubnikov and Kopstik, 1974; Koptsik, 1983; Shubnikov, 1988; Ismael, 1997). The principle is always correct in the phenomena dominated by linear process in practice and therefore a powerful constraint for predicting an unavailable physical condition with perfect symmetry, e.g. in crystallography. On the other hand, several nonlinear phenomena are observed to break the Curie principle (e.g. Radicati, 1987; Stewart and Golubitsky, 1992; Nakamura and Nagahama, 2000). Some of those phenomena have dominant equations described as functional forms. Methods for analyzing the symmetry of a nonlinear system expressed in functional form is mathematically discussed in association with group theory (Sattinger, 1977, 1978, 1979; Fujii and Yamaguti, 1980; Golubitsky and Schaeffer, 1985; Golubitsky *et al.*, 1988). In this paper, we show that breaking of the Curie principle can be mathematically explained for nonlinear functional equations using Sattinger's group theoretical method. This doesn't deny an existence of the Curie principle, rather manifests both preserving and breaking the principle occurs in a nonlinear system.

## 2 NONLINEAR FUNCTIONAL ANALYSIS

Sattinger (1979) described the method to analyze functional in association with group theory. Nonlinear functional equations are generally represented in the form (1)  $F(\mu, w) = 0$  where  $F: \mathbf{R} \times B \rightarrow B$  and  $w = w(\mu)$  ( $\mu \in \mathbf{R}$ ,  $w \in B$ ). It denotes that  $w$  is defined on a Banach space  $B$  and depends on a real parameter (bifurcation parameter) and that  $F$  maps  $w$  to new  $w$  with the change of  $\mu$ . Sattinger investigated the functional equations which mapping  $F$  is covariant under group  $G$ , i.e. (2)  $F(\mu, gw) = gF(\mu, w)$  where  $G$  and  $g$  are set and its element, respectively. In this paper, we consider for unitary group that preserves the inner product invariant. This assumption doesn't lose generality in considering most of real phenomena. Since unitary group is isomorphic with general linear group that is a set of linear mapping, we can take advantage of knowledge and techniques in linear algebra. With application of group representation theory, we can define "standard decomposition" (Sattinger, 1979). We decompose Hilbert space  $H$  here since we have to analyze functional equations on  $H$  to deal with unitary group. Then  $H$  is decomposed into some  $H^\alpha$  which corresponds to each irreducible element  $g_\alpha \in G$  and  $H^\alpha$  intersects orthogonally one another. Therefore, we can now treat variables by its symmetrical compositions, e.g.  $w = (w_1\phi_1, w_2\phi_2, \dots, w_q\phi_q)$  where  $(\phi_1, \phi_2, \dots, \phi_q)$  is a symmetrical basis. Each projection  $P^\alpha$  onto  $H^\alpha$  can also be defined (Sattinger, 1979; Fujii and Yamaguti, 1980). Especially,  $\alpha = 1$  is assigned to isotropy group and  $T_g P^1 = P^1$  for the transformations  $T_g$  ( $g_\alpha \in G$ ). Furthermore, we define Fréchet derivative  $F' =$

$\partial F/\partial w$  to investigate local mapping of functional in the neighborhood of a point  $(\mu_0, w_0)$  at which  $F'$  is defined. If there exists reducible  $F'(\mu_0, w_0)$ , implicit function theorem assures that eq.(1) has a unique solution in the neighborhood of  $(\mu_0, w_0)$ . If  $F$  is irreducible at a certain point, this point is singular and several solutions exist in its neighborhood. For simplicity, we consider about only simple, i.e. not multiple singular points in this paper.

### 3 SYMMETRY IN FUNCTIONAL SYSTEMS

First we consider the case that  $F$  is reducible. Since  $T_g P^1 = P^1$ ,  $T_g w = w$  for  $w = w_1 \in H^1$ . With this  $G$ -invariant property, symmetry preserving of  $G$ -covariant mapping in the subspace is proved. This is to say that for all  $w_1 \in H^1$ ,  $F(\mu, w_1) = P^\alpha F(\mu, w_1)$  is derived from eq.(2). Therefore,  $P^\alpha F(\mu, w_1) = 0$  ( $\alpha = 2, 3, \dots, q$ ). Since  $P^1$  reflects isotropy group, both relations indicate that the solution path of eq.(1) is enclosed in the subspace  $H^1$  unless  $F$  becomes irreducible. In other words, the symmetry of the system does not change on each solution path. Next we consider the case that  $F$  is irreducible. From eq.(2), Fréchet derivative  $F'$  satisfies the following equation  $F'(\mu, T_g w) \cdot T_g = T_g F'(\mu, w)$  ( $\forall g \in G, \forall w \in H$ ). Multiplying by the group characters and summing for  $\forall g \in G$ , we get  $F'(\mu, w_1) P^\alpha = P^\alpha F'(\mu, w_1)$  ( $\alpha = 1, \dots, q; \forall w_1 \in H^1$ ). According to linear algebra, it denotes that  $F'$  can be represented as a diagonal matrix form, i.e.  $F'(\mu, w) = \text{diag}(F'_1, F'_2, \dots, F'_q)$ . Since the basis represents the symmetry, each eigenvalue  $F'_\alpha$  corresponds to a certain symmetrical element of the system. According to Sattinger (1979), the symmetry corresponding  $F'_\alpha = 0$  dominates the new solution path, e.g. the path bifurcating from the singular point. Because the new solution path is more stable (Fujii and Yamaguti, 1980), the system follows the new path. Therefore, if eigenfunction of  $F'_\alpha = 0$  doesn't correspond the symmetry dominated the system until reaching the singular point, the symmetry of the system has not been held yet. Since the old path belongs to the space corresponding the isotropy group, the new path belongs in most case to the space corresponding the other subgroup with the lower symmetry than that of decomposed original space. The system is "dissymmetrical" at the singular point in this sense.

### 4 RECONSIDERATION OF CURIE PRINCIPLE

Finally, we consider the Curie principle in functional analysis. We can derive another Fréchet derivative  $\tilde{F} = \bullet F / \bullet \mu$  in the space with symmetrical basis by the same procedures described above. Because eq.(1) denotes the relation  $w = w(\mu)$ , we can refer  $\mu$  and  $w$  as the cause and the effect, respectively. Then, we can refer  $\tilde{F}$  and  $F'$  as representing the symmetries of the cause and the effect, respectively. Once  $F'_\alpha$  not

corresponding to the existing component of  $F_\alpha$  becomes to 0, the symmetry of the cause is no longer preserved in the symmetry of the effect. This case is just the “breaking of the Curie symmetry principle”. In contrast, even if  $F_\alpha'$  corresponding the existing component of  $F_\alpha$  becomes to 0, the symmetry of the cause is held in the effect. The Curie symmetry principle is preserved in this case. Of course, the principle is preserved if  $F'$  is reducible. In this sense, both preserving and breaking of the Curie symmetry principle occur in a same nonlinear system.

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