

A VISUAL NOTATION FOR RATIONAL NUMBERS *MOD 1*

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Fields of interest: Mathematical visualisation; Visual notation (2D maps, 3D navigable environments); History and philosophy of the scientific image (abstract film and abstract dance, early history of Western musical notation, multimedia as total theatre)

Publications and/or Exhibitions:

Tolmie, J.A. (2000) *Visualisation, Navigation and Mathematical Perception: A Visual Notation for Rational Numbers Mod 1*, [Ph.D. Dissertation], Canberra: Australian National University, 4CDs, 300 animations.

Tolmie, J.A. (2001) *Phase space à côté*, In: *Intersections of Art and Science*, [Exhibition], Sydney: Ivan Dougherty Gallery, COFA, UNSW, [Digital image and animation].

Abstract: *This paper presents a methodology for the construction of visual notation. Visual primitives are chosen. New visual objects are constructed from them. The formalism assumes that once a visual object has been stated, it exists and can be drawn on for use in superposition with, or substitution of, other visual objects. The demonstrated example is rational numbers (or fractions). The torus, with its two directions of rotational symmetry, is used as a phase space; its longitudinal cycle for the denominator; its meridian cycle for the numerator. This results in three-dimensional navigable objects, two of which are discussed. Dominant visual structures are observed in both of these objects. An application of the second visual object orders the bud size of the buds of the main cardioid of the Mandelbrot boundary.*

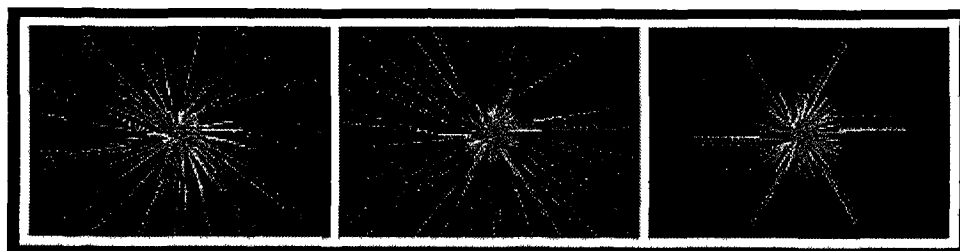
1 VISUAL STRUCTURES IN LIEU OF TEXT

(A visual abstract is provided at <http://www.ozemail.com.au/~jatolmie/isis0.html>.) This work would not exist had it been confined by linear language forms. A visual formalism

is introduced whereby visual objects constructed from previous visual objects and visual primitives substitute for the usual text-based definitions. The challenge was to put rational numbers on stage by choreographing their movements to reveal, or offer insight into, their mathematical structure. To do this involved the questioning of many unwritten conventions. For example, mathematics as a text-based form inherits conventions from natural language (Nordon 1994). As a spatial form, it will inherit conventions from film, theatre, dance, photography, and animation. What are the devices used to punctuate these spaces, and why do they work? (McAuley 1987, Herbison-Evans 1988, Barthes 1964, 1970, 1973, Svoboda 1993, Le Grice 1977.)

In text, mathematical symbols are separated by brackets, commas, dots, operation signs, arrows, etc, with a convention to proceed from left to right. In visual space, visual objects are frequently interwoven. Preferred viewpoints and motions do not yet exist. They are established experimentally, and are to some extent dependent on the type of mathematics notated, and the type of notation used to notate it. While the initial example chosen is geometric in nature (rational numbers *mod* 1 encoded spatially using two coordinates of rotational symmetry), it is the formalisation of pattern and visual objects, and their recognition, which could be applied in a more abstract context.

2 RATIONAL NUMBERS AS CYCLIC CONFIGURATIONS



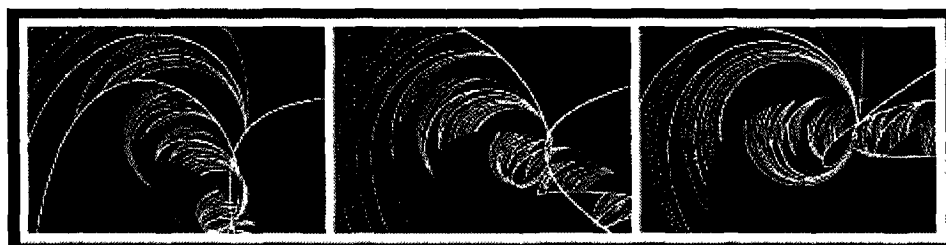
Rational numbers $\mod 1$ are represented abstractly as equivalence classes of pairs (p, q) . But they can also be visualised as parts of a cycle. We would like to use this second notion to create visual symbols for them. Firstly, the denominator of a rational number is encoded as radial direction (in the horizontal plane). Then, using cyclic motion and a discrete map of $\{1, 2, 3, \dots, 37\}$ into the RGB colour space, the numerator is encoded as cyclic permutations of colour coded dots. These configurations are placed in the appropriate meridian slices (in radial vertical planes). The result is a three dimensional navigable object contained in a torus. In three dimensions, colour alignments and relative distributions of elements are the most striking aspects to the space. In two dimensions, sequences of rational numbers are viewed as abstract animation. Dominant visual structures are perceived dynamically. Radial directions behave simultaneously as sources

and sinks. The subpatterns of convergence are analysed by defining each radial direction as a sequence in time. A particular radial direction either appears or does not appear in an individual frame of the animation. Tracing the subpatterns backwards reveals the Farey tree structure of the rational numbers *mod* 1.

3 RATIONAL NUMBERS AS FAREY CURVE SEGMENTS

The Farey tree is a binary tree construction of the rational numbers. (see Bogomolny <http://www.cut-the-knot.com/blue/Farey.html>). To introduce rotational symmetry, the Farey tree is made circular and embedded in the longitudinal plane of the torus. Individual rationals are then defined as curve segments which span the longitudinal region bounded by its Farey parents. These curve segments are cut from curves which wind around the torus at a constant integer velocity (integer velocity one parameter subgroups of the torus as a Lie group). A three dimensional navigable object is made.

The dominant visual structures are “fan” like objects which emanate from rational lattice points above and below the outer longitude of the torus. The rational numbers *mod* 1 are then ordered by these fans, providing insight into their tidal interweaving.



4 APPLICATION TO MANDELBROT SET – ORDERING OF BUD SIZE

There is an immediate application: The buds of the main cardioid of the Mandelbrot boundary can be put into one to one correspondence with the rational numbers *mod* 1. We “forget” its continuous structure. We are only interested in the ordering of the buds, by size, in sequences of buds, and in the interweaving of these sequences. It turns out that the above ordering of rational numbers *mod* 1 by fans induces a natural ordering on the bud size of sequences of buds of the main cardioid of the Mandelbrot boundary.

A visual notation is created by using a two dimensional fan to denote a three dimensional fan. The circular Farey tree is embedded in the unit circle and fans are attached to its vertices with endpoints the rational points on the circle. A single vertex in the tree now represents an infinite sequence of rational numbers. In this way, the internal Farey trees acts as an indexing set for the tiling of the Mandelbrot boundary by infinite sequences of buds (fans). In other words, the interwoven extended structure on the Mandelbrot boundary is disentangled by mapping it to distinct initial points in the Farey tree. In this representation, individual buds do not exist. The primitive unit is the sequence of buds. As such, the fans on the torus provide a three dimensional phase space for viewing the discrete visual structure of the Mandelbrot boundary. (See on the front page of this Journal.)

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