

GEOMETRY OF DEMOCRACY

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Abstract: *one-person, one vote is a basic principle of democracy. What is the fairest allocation system of the seats to parties in election? It is an interesting mathematical problem. Proportional representation is introduced in many countries. It is applied for allocation of the seats sometimes to parties and sometimes to states. There are essentially two allocation schemes of seats, by D'Hondt and by Sainte Lagües. Here a new interpretation of these schemes is proposed and geometrically analyzed. How does fairness look as geometrical feature? What is the shape in Euclidean space? The main part of the presentation is the geometrical part.*

1. Proportional Allocation in Electoral System

The problem of allocation in electoral system is to find the proper set of the same number of integers as that of parties; the sum of them is a given number and the ratio of them is as close to the percentage of votes as possible. The scheme by D'Hondt is conventionally explained with its allocation procedure as follows.

Suppose there are k parties $(1, 2, \dots, k)$, their number of votes are respectively V_i

$(i=1, 2, \dots, k)$, the total number of votes is $V = \sum_{i=1}^k V_i$, and the relative votes are

$v_i \equiv V_i/V$. N_i seats among N seats are allocated to i -th party so that $N = \sum_{i=1}^k N_i$.

The conventional explanation of the scheme is based on the practical procedure of the allocation. It is summarized as follows. First, make a table of the number of votes divided by positive integers party by party. Next, make the list of the largest N numbers among the table. Party i get N_i seats if N_i numbers are included in the list of N numbers just mentioned.

The hidden principle of the D'Hondt scheme, which the above procedure based on, is to

keep the balance in the number of the votes per a seat between the parties. It is formulated in a mathematical expression: the following relation must hold between any two parties i and j .

$$\frac{V_i}{N_i + 1} < \frac{V_j}{N_j} \text{ and } \frac{V_i}{N_i} > \frac{V_j}{N_j + 1}.$$

New interpretation can be simply understood. Some proper number L can be found so

$$\text{that } N_i < n_i L < N_i + 1 \text{ and } N = \sum_{i=1}^k N_i$$

Generally, the number L lies in some range. The only exception is accidental degeneracy. For example, any scheme has nothing to do if the number of votes of some different parties is the same and they are at the critical line.

It is noted that the expression in terms of N/V is more easy to put some meaning than those in terms of V/N at least in the above mentioned new interpretation and the next example. The practical procedure in the scheme by Sainte Lagües is similar to the D'Hondt's one except that the divisor is restricted to odd numbers in making table of V/N . Some textbook describes that the base of the scheme is unclear. But the base is clear. There are two representative methods in making an integer from some real number, round down and round off. The former corresponds to the D'Hondt's and the latter to the Sainte Lagües'. It is easily seen in terms of N/V since the description of rounding off is easier to see.

2. Geometrical Analysis (The case of three parties.)

The case of just three parties is a good example since it can be shown in some planer figures. After an election, the initial data is the number or the percentage of the votes to every party. It can be expressed as a point in a regular triangle since the sum of the lengths of three perpendiculars to edge is common to all points in it.

An allocation scheme decides a tessellation of the triangle for every given integer so that a polygonal domain corresponds to a set of three integers. The common feature of proportional schemes in this stage is that all the boundary lines of the domains are a part of line radiating out from one of vertices of the triangle. The radiating nature is one of geometrical characteristics of proportionality. It is noted that the case of just two parties appears at an edge here. The radiating nature strictly keeps the allocation to two parties even the third party exists. The three parties in the story up to here can be extended almost in parallel to the case of any three among many. What should be kept in mind is only that the number of available seats for the three depends on the other parts. Such independent natures are also character of proportional allocation.

It is noted that all the domains in Fig. 1 have the same area, irrespective of the shape, square, pentagon, and hexagon. The areas of the domains in Fig. 2 are three kinds depending on square, pentagon, or hexagon.

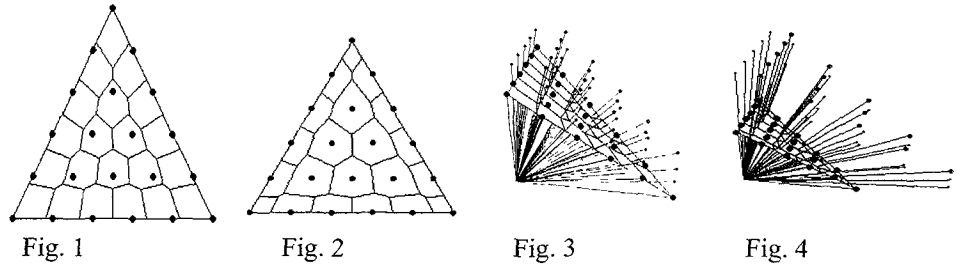


Fig.1 Domains for 5-seats 3-parties case in D'Hondt scheme. Black circles are lattice points.
 Fig.2 Domains for 5-seats 3-parties case in Sainte Lagüe scheme.
 Fig. 3. D'Hondt's case. Any vertex of the domains in Fig. 1 is on the line connecting a certain lattice point and the origin in 3D.
 Fig. 4. Any vertex of the domain in Fig. 2 (Sainte Lagues' case) is on the line connecting a certain lattice point and the origin in 3D. The lattice points are more distant than D'Hondt's case in Fig. 3.

3. General case with more parties

The allocation of seats consists a high dimensional lattice, the dimensionality of which is the number of the parties. The problem is to find the proper system to find the most proper lattice point for any distribution of votes. A system divides the space into domains of lattice points. The borders of such domains direct to the corner of the space in many respects.

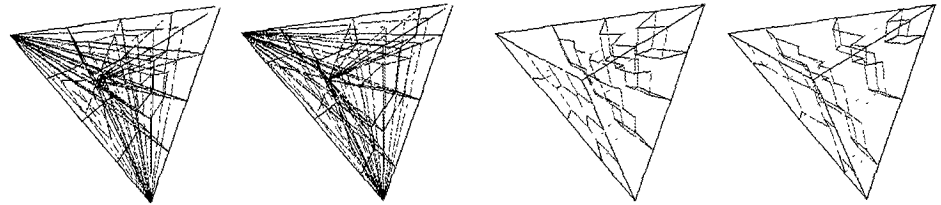


Fig. 5 The case of four parties shown as tessellation of tetrahedron. Left two shows radiating nature and right two the wire frame structure corresponding to Fig. 1 and Fig 2 of 3 parties case. The first and the third from left are D'Hondt case and the others are Sainte Lagues case.

References

Tohru Ogawa and Taeko Ogawa; “One-Dimensional Quasicrystals and Democracy”, in *Quasicrystals* (Eds. K. H. Kuo and T. Ninomiya, World Scientific, 1991) pp.394-401.