Symmetry: Art and Science
Sydney Congress, 2001

POLYGONS AND CHAOS

JAY KAPPRAFF and GARY W. ADAMSON

Name: Jay Kappraff *Address:* Department of Mathematics, New Jersey Institute of Technology, U.S.A.
E-mail: Kappraff@aol.com
Name: Gary W. Adamson *Address:* P.O. Box 124571, San Diego, CA 92112-4571, U.S.A.

Abstract: *An infinite number of periodic trajectories are derived for the logistic equation of dynamic systems theory at a value of the parameter corresponding to the extreme point on the real axis of the Mandelbrot set. Beginning with the edge of a family of star* n*-gons as the seed, the trajectory of the logistic map cycles through a sequence of edges of other star* n*-gons. Each* n*-gon for n odd is shown to have its own characteristic cycle length. The logistic map is shown to be one of an infinite families of maps, all exhibiting periodic trajectories, derived from a family pf polynomials related to the Lucas sequence.*

1 The Relationship between Polygons and Chaos for the Cyclotomic 7-gon

Consider a sequence of polynomials (see Table 1) whose coefficients, disregarding signs, sum to the Lucas sequence: 1, 3, 4, 7, 11, ..., a Fibonacci-type sequence.

Table 1. The Lucas Polynomials

L_1	x	1	L_5	x^5-5x^3+5x	11
L_2	x^2-2	3	L_6	$x^6-6x^4-9x^2-2$	18
L_3	x^3-3x	4	L_7	$x^7-7x^5+14x^3-7x$	29
L_4	x^4-4x^2+2	7		...	

These polynomials are generated, starting with 2 and x, by the recursive formul $L_{n+1} = xL_n - L_{n-1}$ and exhibit the crucial property,

$$L_m(2\cos\theta) = 2\cos m\theta. \qquad (1)$$

Consider the second Lucas polynomial L_2,, $x^2 - 2$ and its iterative map,

$$x \mapsto x^2 - 2, \qquad (2)$$

also known as the *logistic map*. It represents the extreme left-hand point on the real axis of the *Mandelbrot set at the onset of full-blown chaos in which* its Julia set is disconnected, comprising a Cantor set (Schroeder, 1990, Peitgen 1992).

Starting with a seed value x_0 and placing it into Equation 2, the sequence $x_0, x_1, x_2, \ldots$ is generated and referred to as the *trajectory* of the map. If $x_n = x_0$, the trajectory repeats and is said to be an n-cycle. As a result of our analysis, there exist cycles of all lengths which can be characterized as edges of star $2n$-gons for n odd in which each value of n has its own characteristic cycle length.

2 Star Polygons and the Cyclotomic n-gon

Consider $2\cos\dfrac{2k\pi}{n}$ for $k = 1,2,3,\ldots, \dfrac{n-1}{2}$ and n odd, ignoring the signs. These expressions are the real parts of n-th roots of unity given by the equation,

$z^n - 1 = 0$ for z a complex number and n an odd integer.

The complex roots of this equation form a regular n-gon of unit radius known as an n-cyclotomic polygon (Kappraff, 2001). It can be shown that for arbitrary k-values, there is a j-value such that,

$$2\cos\frac{2k\pi}{n} = 2\sin\frac{j\pi}{2n} \quad \text{where } 4k + j = n. \qquad (3)$$

Furthermore it can be shown that, corresponding to an n-cyclotomic polygon for n odd, $2\sin\dfrac{j\pi}{2n}$ is the length of the j-th diagonal (including the edge, j=1) of a regular $2n$-gon of radius 1 (Kappraff, 2002). Also when j is relatively prime to $2n$ the diagonal can be thought of as the edge of the star n-gon $\{2n/j\}$ for j>0 and $\{2n/2n+j\}$ for j<0. Therefore according to Equations 1 and 3, an edge of a star $2n$-gon maps to the edge of another star $2n$-gon.

3 Polygons and Chaos for the Cyclotomic 7-gon

Consider the cyclotomic 7-gon. Beginning with a seed value of, if $x_0 = 2\cos\dfrac{2\pi}{7}$, the iterates are the sequence of edge lengths of different species of star 14-gon corresponding to

$$2\cos\frac{2k\pi}{7} \text{ for, } \quad k \equiv 1,2,4,8,\ldots(\text{mod } 7). \qquad (4)$$

Since $8 \equiv 1$ (mod 7) the sequence repeats with the 3-cycle,

$$x_0 -> x_1 -> x_2 -> x_0, \text{ or } \quad 2\cos\frac{2\pi}{7} = 1.2469\ldots -> 2\cos\frac{4\pi}{7} = -0.44509\ldots$$

$$-> 2\cos\frac{8\pi}{7} = -1.80189\ldots -> 2\cos\frac{16\pi}{7} = 1.2469\ldots$$

Since $\cos\frac{2\pi\,(n-k)}{n} = \cos\frac{2k\pi}{n}$ positive and negative values of the index k (mod n) correspond to identical edge lengths. Therefore 4 (mod 7) $\equiv -3$ is equivalent to $k = 3$ in Sequence 4, and so the 3-cycle is represented by the sequence of k-values: 1->2->3->1... The corresponding sequence of j-values , according to equation 3, is: 3-->-1-->-5-->3. These are listed in Table 2 along with the corresponding sequence of star 14-gons: {14/3}-->{14/13}-->{14/9}-->{14/3} also denoted by the sequence: 3->13->9->3... If the vertices of the polygon are numbered from 0 to 13 this sequence can be associated with the edges of star 14-gons drawn from vertex number 0 of the 14-gon to a vertex of this sequence. The *order* of the edges in the cycle are also indicated in Table 2 beginning with the seed $i = 0$. Results for the cyclotomic 9-, 11-,13- and 17-gons and a general algorithm for determining the cycle of any cyclotomic n-gon can be found in (Kappraff, 2001).

Table 2. Cycles for the Logistic Equation Corresponding to the Cyclotomic 7-gon.

k	$2\cos\frac{2\pi k}{7}$	$2\sin\frac{\pi j}{14}$ j	{14/j} or {14/14+j}	Order i
1	1.24696...	3	3	0
2	0.44509...	-1	13	1
3	1.80189...	-5	9	2

4 Polygons and Chaos for Generalized Logistic Equations

We have found that the entire sequence of Lucas polynomials $L_m(x)$ in Table 1 represent generalized "logistic" maps exhibiting cycles due to their property shown in Equation 1 which states that an edge of a star $2n$-gon maps to the edge of another star $2n$-gon.

We have also proven that the cycle length corresponding to any cyclotomic n-gon for n odd is equal to the smallest exponent p such that,

$$(m^2)^p \equiv 1 \pmod{n} \qquad (5)$$

Where m is the index of the m-th Lucas polynomial. Values of p are listed in Table 3 for values of $n = 7,9,11,13,17$ and $m = 2,3,4,5$.

Table 3 Exponents p such that $(m^2)^p \equiv 1 \pmod{n}$

m^2	$N = 7$	$n = 9$	$n = 11$	$n = 13$	$n = 15$	$N = 17$
4	3	3	5	6	4	4
9	3	...	5	3	...	8
16	3	3	5	3	2	2
25	3	3	5	2	...	8

From this table we see that the cyclotomic 7-gon has a 3-cycle for $m = 2,3,4$, and 5 as described above for $m = 2$. Notice that the 9-gon has only a 3-cycle corresponding to $2\cos\frac{2k\pi}{n}$ for $k = 1,2,4 = \frac{9-1}{2}$ with $k = 3$ missing since 3 and 9 are not relatively prime. We have also found that the period length is a divisor (or factor) of the number of integers $1,2,3,\ldots, \frac{n-1}{2}$ relatively prime to n. For example, for a cyclotomic 15-gon the only integers from 1,2,...,7 that are relatively prime to 15 are 1,2,4,7 so that periods for the cyclotomic 15-gon must be factors of 4. We see from Table 3 that periods of 2 and 4 occur for the L_2 and L_4 Logistic Maps.

5 Conclusion

We have shown that at a critical point of the Mandelbrot set where orbits of the logistic equation begin to escape, each of these periods can be characterized by a sequence of edge lengths of a family of star $2n$-gons, for odd n, each n having a characteristic cycle length.

References

Kappraff, J. (2001) Polygons and chaos, In: Jablan, S., ed., *Bridges*, Winfield: Great Plains Press.

Kappraff, J. (2002) *Beyond Measure: A Guided Tour through Nature, Myth, and Number*, Singapore: World Scientific Publ., In press.

Peitgen, H.-O., Jurgens, H., and Saupe, D. (1992) *Chaos and Fractals*, New York: Springer-Verlag.

Schroeder, M. (1990) *Fractals, Chaos, Power Laws*, New York: W.H. Freman Press, 1990.