

SYMMETRY: SCIENCE AND ART

**THE EMERGENCE OF SYMMETRY CONCEPTS BY
THE WAY OF THE STUDY OF CRYSTALS (1600-1900)**

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INTRODUCTION

The word symmetry has, in the dictionaries, two types of definition. The first corresponds to the ideas of “harmony” or “just or due proportion.” In the beginning of the period that we are studying, this was the only meaning of the word and it was used by artists. The second meaning corresponds to set of isometries that are geometrical motions such as rotations, translations and reflexions. This last mathematical meaning was issued from the first one during these three hundred years. The most important period for this transformation was the beginning of the 19th century. The choice of 1900 to close our historical description is due to two important scientific results published between 1890 and 1900: E. Fedorov (1890) and A. Schoenflies (1891) published almost independently the list of the 230 “space groups of symmetry” which are a model of formalization of symmetry properties of objects in our three dimensional space. Then in 1894, P. Curie introduced symmetry in physics in a famous paper. The 20th century would then be the century of the use of symmetry concept in many branches of modern science.

During three centuries, one of the essential ways which allowed the transformation from the original sense of harmony to that of set of certain geometrical transformations was the study of crystals.

1 OBSERVATION OF CRYSTALS

Among many curious features presented by nature, crystals with their planar faces limited by rectilinear edges, held the scientists' attention. Some of them which will be presented here attempted to establish a relation between the crystal shape and a hypothetical microscopic model of solid matter: shapes and organization of minute particles could explain the observed macroscopic shape of crystals characterized by certain regularities.

This was not the only question formulated by these scientists, but it was a central one. It was also common with the scientists who developed theories of structure of matter during 17th, 18th and 19th century from which theory of symmetry was an essential component, as will be seen subsequently.

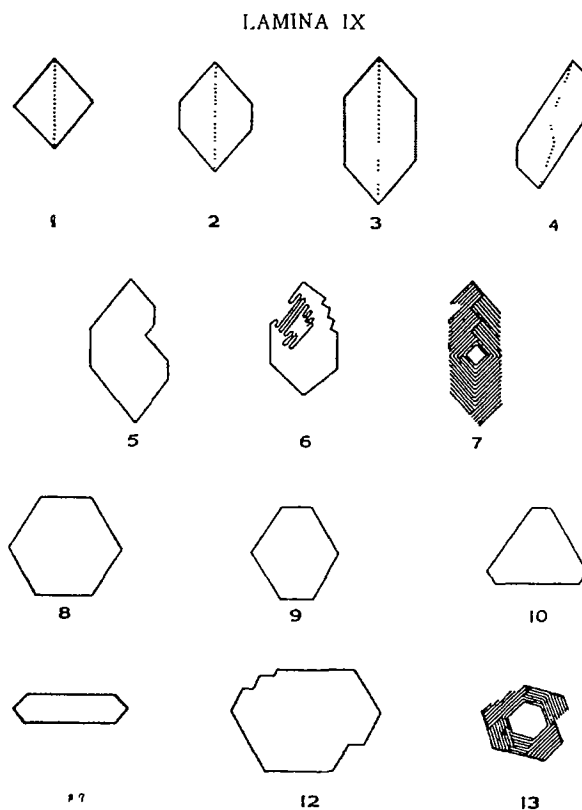


Figure 1: Plate n° IX of the Prodomus by N. Steno

The first usually cited scientist is Johann Kepler. In his small book *Strena Seu de Nive Sexangula* (1611), he presented observations of snowflakes with their characteristic angle of sixty degrees between rods that come from a common center. He studied the different possible packing of minute spheres of ice in an attempt to explain the hexagonal regularity observed on snowflakes. He did not succeed in explaining the shapes of snowflakes on a purely geometric basis. But its intellectual process would prove fruitful in the future.

In 1669 Nicolas Steno published the summary of an ambitious opus in which, among other things, he drew sections of quartz crystals cut up into different directions, and he measured the angles of the polygons of these figures. He pointed out the constancy of these angles in the case of quartz (Figure 1).

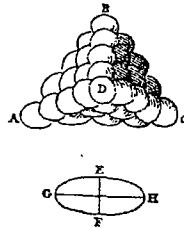
In the same year, 1669, Erasmus Bartholinus published a paper dedicated to Iceland spar: some beautiful crystals of calcite had been brought back from this island the year before. He discovered the double refraction of light and named the two types of rays ordinary and extraordinary. First, he observed the rhombohedral shape of these crystals and also that cleavage retained this form. He characterized this geometrical form by measurements of ordinary and dihedral angles.

Christiaan Huyghens took up again this study in his *Traité de la Lumière* published in 1690. The double refraction of Iceland spar he analyzed and explained allowed him to propose his theory of light as a vibration. He described more precisely the rhombohedron shape of these crystals and proposed a model of *ellipsoidal* particles as constituents (Figure 2). Then he could explain the external shape and cleavage properties. This very deep intuition had been understood and supported only 150 years later.

These examples, taken out among many others as R. Hooke, D. Guglielmini, show us the process and the results obtained on these subjects in the beginning of the 18th century: geometrical regularities (such as angles of 60°, 90°) were observed on minerals. However attempts to explain these regularities from microscopic theories of structure of solid matter failed.

Nevertheless, from our point of view, i.e., the progressive elaboration of symmetry concepts, the regularities observed on the shape of different crystals constitute an experimental fact which will be used later in the definition of “homologues” or “identical” parts of crystal as edges or vertices. And then, it constitutes an important step in this history.

sels, et de celle du sucre, l'on trouve d'autres angles solides, avec des surfaces parfaitement plates. La neige menue tombe presque tous-jours formée en petites estoiles à 6 pointes, et quelques fois en hexagones dont les costez sont droits. Et j'ay souvent observé, au dedans de l'eau qui commence à se geler, une maniere de feuilles



plattes et deliées de glace, dont la raye du milieu jette des branches inclinées d'un angle de 60 degrez. Toutes ces choses meritent d'estre recherchées soigneusement, pour reconnoitre comment et par quel artifice la nature y opere. Mais ce n'est pas maintenant mon dessein de traiter entierement cette matiere. Il semble qu'en general la regularité, qui se trouve dans ces productions, vient

de l'arrangement des petites particules invisibles et égales dont elles sont composées. Et pour venir à nostre Cristal d'Islande, je dis que s'il y avoit une pyramide comme ABCD, composée de petits corpuscules ronds, non pas spheriques, mais spheroides plats, tels que se feroient par la conversion de cette ellipse GH sur son petit diametre EF; dont la proportion au grand est fort près celle de 1 à la racine quarrée de 8. Je dis donc que l'angle solide de la pointe D, seroit égal à l'angle obtus et equilateral de ce Cristal. Je dis de plus, si

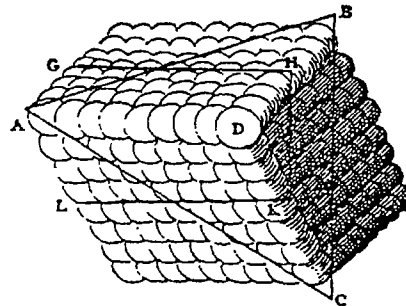


Figure 2: Extract from the *Traité de la Lumière* by C. Huyghens

Throughout the 18th century, almost up to its end, the type of questioning of Kepler or Huyghens upon crystals was given up. Using our categories, one can say that Physics and Natural sciences were split off. Naturalists who studied minerals were interested essentially in the questions of their classification and of their origin. As regards classification, it seemed to many people that the external shape of crystals was not a relevant criterion: one mineral as calcite for example may take on various shapes; different minerals appear to have the same shape, the cubic one, for example, as rock salt and pyrites. Buffon can be held up as an example of this position.

In contrast, C. Linneaus reproduced the shapes of about forty minerals in the plates of his *Systema Naturae* (1768). It demonstrates he thought that shape could be an interesting feature. Thus, he showed the way to J. B. Romé de Lisle who accomplished a decisive step in the observation of the geometrical regularities of the shapes of crystals.

2 T. BERGMAN, J-B ROME DE LISLE, R-J HAÛY. BIRTH OF CRYSTALLOGRAPHY

The word crystallography was introduced by M. Cappeller in the beginning of the 18th century. But the crystallographic science was made possible only when a microscopic theory of structure was postulated the predictions of which could be compared with experiments. This foundation can be attributed to R-J Haüy. However, the role of two precursors must be described before.

J-B Romé de Lisle established the catalogues of several mineralogic collections and then wrote *Essai de Cristallographie* (1772) which was revised and published under the title *Cristallographie* in 1783. The number of minerals described in this last book reaches 400. Two important ideas must be pointed out. The first one is the process thanks to which a crystal “habit” can be described: a “primitive form” is truncated on its edges or on its vertices by little planes in order to obtain the observed crystal shape. This process is seen as an intellectual operation and not as an operation of Nature. Classification of crystals based on the symmetry of their primitive form thus became possible. The attempts of Romé de Lisle to classify crystals in this way show some errors. The second essential point included in the first law of crystallography, named the law of constancy of interfacial angles. It appears in his last book. This law extends the observations of Steno or Huyghens made on certain crystal species. It says that dihedral angles between crystal faces are the only pertinent parameters allowing a quantitative description of the external forms of a crystalline species. Areas or shapes of crystal faces are not such parameters.

One can remember that A. Carangeot, Romé’s assistant who had to make crystal models drew the first *goniometer* and observed the equality of dihedral angles measured on real crystals. Then Romé verified this fact and could publish the law under a generalized form. It can be noticed that Romé did not want to build a theory, because he thought that it would be too hypothetical at this time.

Torbem Bergman, a Swedish mineralogist and chemist, explored the idea of determining the primitive form of crystals, using cleavage. He could prove than one habit of calcite, the scalenohedron, was then related to another, the obtuse rhombohedron. But, his attempts to generalize this approach failed off and he abandoned this research.

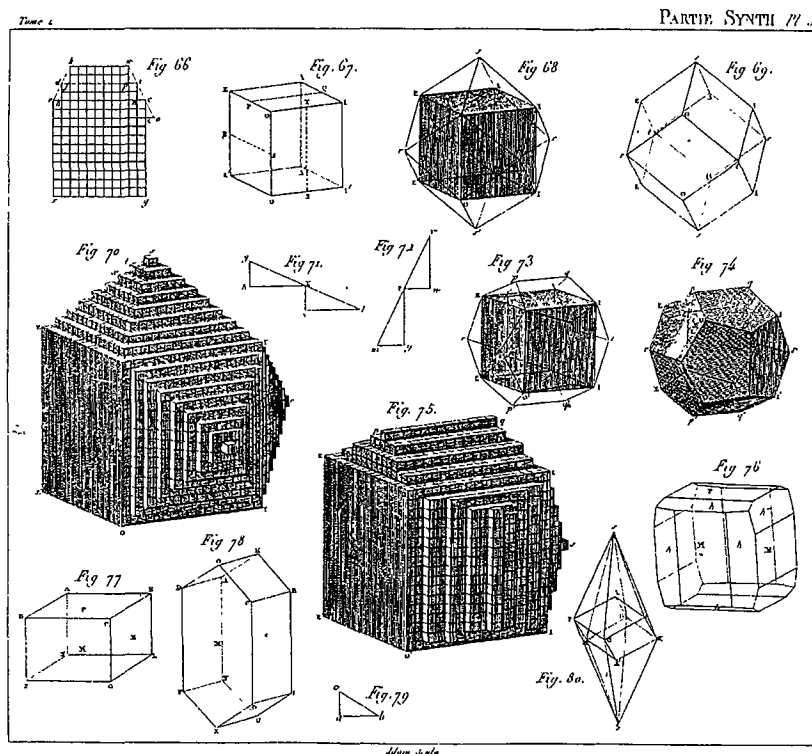
All the credit of the establishment of an ambitious and general theory of crystal structure is due to R-J. Haüy which is thus named “the father of crystallography.”

The bases of his theory were laid out as early as 1781, 1782; a first synthesis was published in 1784 and until his death in 1822, this theory did not evolve in a significant manner. The easy cleavages of a crystal of any shape allow to obtain the primitive form of its nucleus, a polyhedral form which corresponds to that of the chemical unit of this crystal, named *molécule intégrante*. These building blocks of the crystal present the same polyhedral shape but they are not visible to the naked eye, due to their small size. Juxtaposition of these small units permits to obtain the real macroscopic form of the crystal: the crystal faces are smooth or stepped according to their own “law of decrement” which can be characterized by integer numbers, characteristic of each face.

This model was illustrated by figures as that of Figure 3, which played an important role in the spreading and understanding of this theoretical model. The important point is the possibility of comparison between the experiment – precise geometric characterization of crystal shape – and the theory – hypothetical form of the polyhedron of the *molécule intégrante*.

At the origin of substantial progress in physics, mathematics and chemistry, this theory suffered from some shortcomings. Two of these will be enunciated here: the possibility for the primitive form to be an octahedron or a tetrahedron leads to difficulties in order to fill space. Then, the theory was complicated by the introduction of the *molécules soustractives*. The consistency of the theory was significantly reduced. A second difficulty arose with the identity claimed between geometrical unit and chemical unit. Phenomena such as polymorphism and isomorphism recognized during Haüy’s life were inconsistent with this element of the theory.

Nevertheless, Haüy introduced in the study of crystals the relevant part of mathematics – geometry in three-dimensional space – which will be used as the language and the frame to test structural theories. It was the law of constancy of dihedral angles which enabled to submit hypothetical geometric constructions to critical test. Mathematical concepts of symmetry were born from this experience.



3 R-J HAÜY – FROM IMPLICIT TO EXPLICIT SYMMETRY

The scientific production of Haüy consists in some books and more than 100 articles, most of them corresponding to the description of mineral species. One can follow the evolution regarding the general concept of symmetry. From the beginning of the nineties, remarks such as the following ones can be read (Haüy, 1796a):

“Le calcul théorique fait voir, de plus, que chaque rhombe c d e q, g o n y, &c. est semblable au rhombe primitif du spath calcaire, et que l’angle obtus e g y de chaque trapèze est égal à chacun des angles e d L, 9 o I, &c. de l’octogone voisin, c’est-à-dire, qu’il est de $116^{\text{d}} 33' 55''$. On a pu remarquer encore dans ce qui précède, que l’inclinaison des mêmes trapèzes sur les pans adjacens était égale à celle que les pans gardent entre eux, c’est-à-dire de 120^{d} . Toute la théorie est pleine de ces analogies et de ces propriétés géométriques, qui répandent une sorte d’harmonie dans les résultats des lois auxquelles est soumise la structure des cristaux.” (Figure 4).

The last sentence could be translated as follows:

“All the theory is full of these analogies and of these geometrical properties which spread a sort of harmony into the results of the laws which the structure of crystal obeys.”

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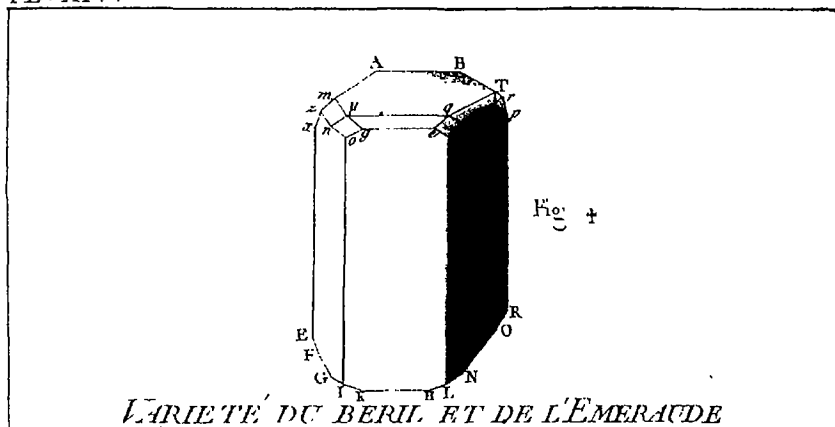


Figure 4: Figure of the note on crystallisation of emerald (Haüy 1796a)

In a paper published slightly later, Haüy (1796b) used properties of symmetry without any precise characterization. In order to establish a nomenclature of vertices, edges and faces of polyhedron representing crystals, he named these elements of figure by letters (Figure 5). He took into account the proper symmetry of each polyhedron and then used a minimal number of different letters. In the description of the homology between the edges (or vertices or faces) of regular polyhedra, one can see an implicit knowledge of their characteristic symmetry (with several errors).

This implicit use of symmetry rules would continue up to the publication in 1815 of a paper entitled “*Mémoire sur une loi de cristallisation appelée Loi de Symétrie*” (Memoir on one law of crystallisation named Law of Symmetry). In this paper, Haüy related the number and the position of the faces observed on the external form of crystals to the symmetry of the hypothetical nucleus, the *molécule intégrante*. This last symmetry is considered obvious and does not need to be described: in a cube, all the vertices are equivalent (Haüy writes identical), and so are the edges and the faces. In a square-based prism, only 4 faces are equivalent and the 2 others only between themselves, etc.

It must be emphasized that the word symmetry here possesses the new acception (set of isometries), but that the tools necessary to describe these isometries do not exist. Nevertheless, the concept of the homologous parts of a figure has been found out, and it will be used as a pathway to transform this qualitative concept to quantitative ones.

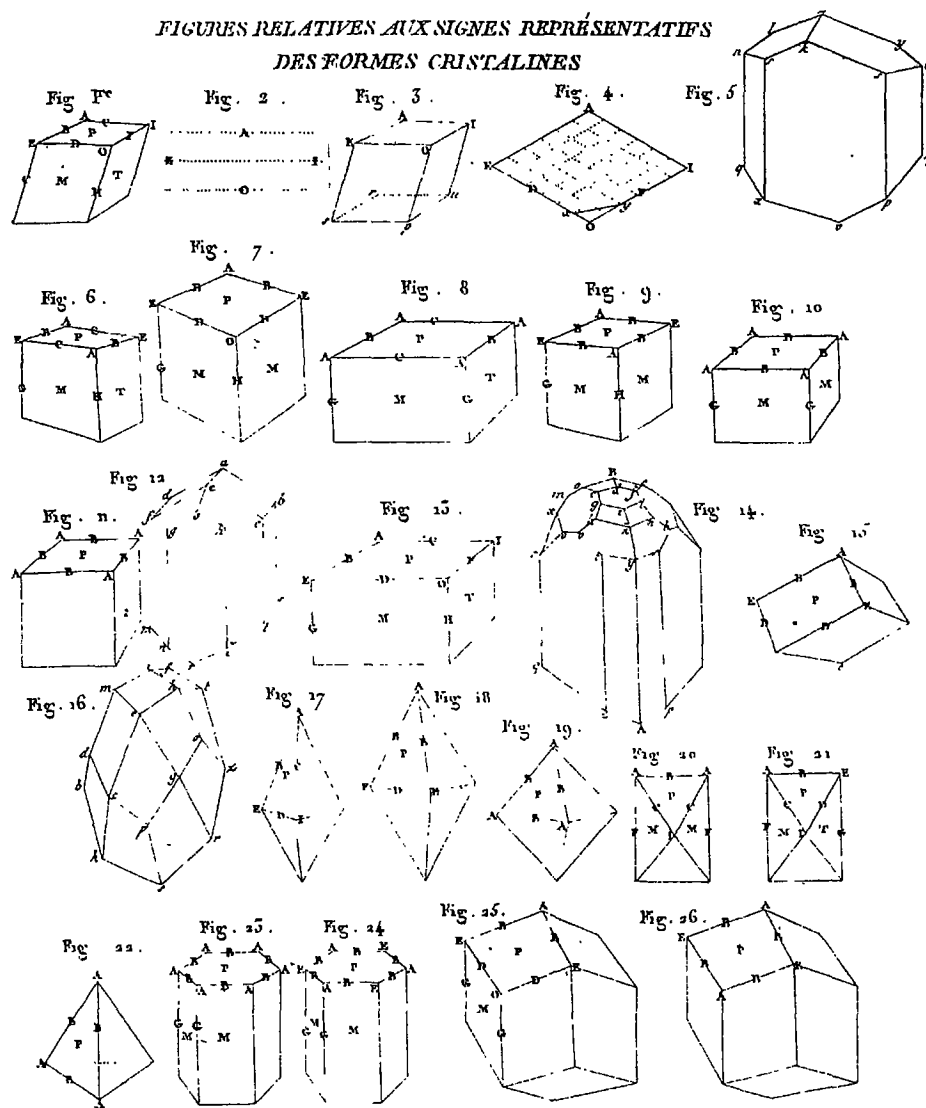


Figure 5: Figure of the "Exposé d'une méthode simple et facile..." (Haüy 1796b)

4 FROM S. C. WEISS TO A. BRAVAIS

The great importance of Haüy's Law of Symmetry is due to the connection he establishes between the obvious symmetry of the microscopic nucleus and that of the shapes of real crystals. This would lead to the great development of studies and classifications of the external symmetries of crystals. It will be the work of S. C. Weiss a German crystallographer, a disciple of Werner and of Haüy and that of his pupils, G. Rose, F. E. Neumann, F. Mohs, ... The notion of the crystal system based on symmetry emerged during the period 1815-1830. The description of the 32 crystal classes was published by L. M. Frankenheim in 1826 and independently by J. F. C. Hessel in 1830. Neither had any influence on the science of the time.

The 32 crystal classes correspond to the combinations of the following symmetry elements: the rotation axis of order 2, 3, 4 and 6, the inversion center and the mirror plane. The existence of these classes is proved by the possible coexistence of one of these symmetry elements with one or more others.

This type of purely mathematical analysis must enable the crystallographer to divide real crystals among the 32 classes, each crystal corresponding to a unique class.

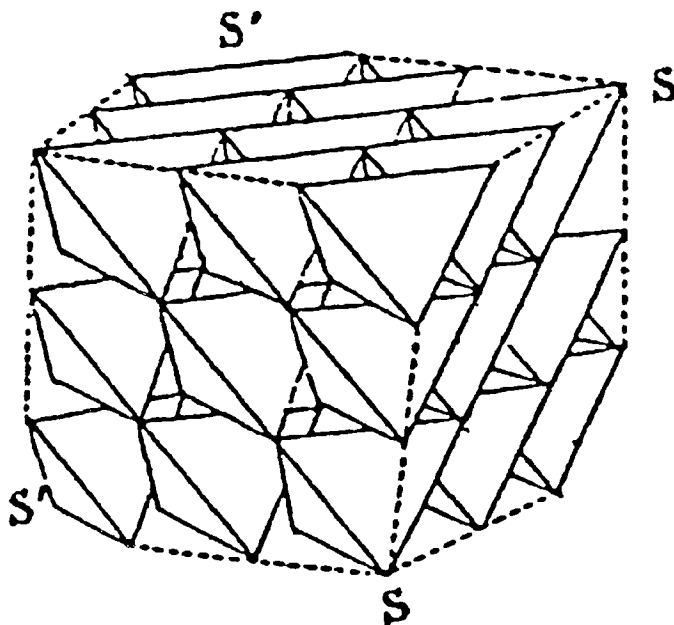


Figure 6: Model proposed by G. Delafosse for the boracite crystal (Delafosse 1843)

Another way conducted A. Bravais to the same result. As Frankenheim and other scientists (Wollaston, Seeber, for example), G. Delafosse wanted to improve the microscopic structural model of crystalline matter proposed by Haüy. He assumed that the building blocks of matter, each separate from one another (and not contiguously joined as in Haüy's model) consisted into polyhedral molecules similarly oriented at the nodes of a three dimensional lattice. Figure 6 gives an example of tetrahedral molecules situated on a cubic lattice. One more time, this type of representation would substantially help the diffusion of the model which was revealed as fruitful. L. Pasteur and A. Bravais used this model for their own discoveries. It may be noted that Delafosse does not say that molecules are polyhedra, but they do possess the symmetry of a polyhedron.

In order to analyse completely the symmetry of crystals, from this theoretical point of view, A. Bravais studied separately the 14 possible symmetries of lattices (regular system of points, 1849) and the 23 possible symmetries of polyhedra (1850) (Table 1). In 1834, Frankenheim had stated without demonstration the existence of 15 (which are actually 14) space lattices. Besides, one class of polyhedra was omitted by Bravais in his paper, but it was reintegrated in his Crystallographic Studies of 1851 when he applied his mathematical results to that study of real crystals where he gave out the existence of the 32 crystal classes (crystal point groups nowadays).

At the crucial stage of our history, it may be useful to do some comments. Bravais was a remarkable mineralogist and crystallographer as well a rigorous mathematician, and later, he analysed the very qualitative model of Delafosse with the relevant geometrical tool. The rigour and clarity of his papers ensured that they were read up and therefore they had a numerous posterity in mathematics as well as in crystallography. The 14 "Bravais lattices", the 23 symmetry classes of polyhedra or the 32 crystallographic point groups are known nowadays as groups in the mathematical sense of this word. They were groups before the existence of groups. One can note that the notion of the symmetry element is present but not that of the symmetry operation. Likewise Bravais studied the coexistence of symmetry elements, their combination but the idea of a composition law of these operations was not yet present.

Known for more than two thousand years, the five Platonic solids fascinated artists and mathematicians for their "regularity", i.e., the equality between faces and between edges and for their limited number. Nowadays, these polyhedra are seen as archetypes of symmetry groups. But virtue of hindsight, it may be considered curious that their essential characteristic, from our point of view, was not underlined before 1850. One can think that experiments carried out on less regular polyhedra that are real crystals have permitted to understand what symmetry is: First, there are degrees into symmetry, and second, the symmetry of an object can be decomposed into elementary operations.

Classification des polyèdres, d'après la nature de leur symétrie.

POLYÈDRE		SYMBOLE DE LA SYMÉTRIE DU POLYÈDRE.	CLASSE du polyèdre	
Asymétrique.....		$\circ L, \circ C, \circ P.$	1 ^{re}	
Symétrique	dépourvu d'axes.....	$\circ L, C, \circ P.$	2 ^e	
		$\circ L, \circ C, P.$	3 ^e	
	pourvu d'un axe principal	d'ordre pair...	$\Lambda^2 q, \circ L^2, \circ C, \circ P.$	4 ^e
			$\Lambda^2 q, \circ L^2, C, \Pi.$	5 ^e
			$\Lambda^2 q, q L^2, q L'^2, \circ C, \circ P.$	6 ^e
			$\Lambda^2 q, \circ L^2, \circ C, q P, q P'.$	7 ^e
			$\Lambda^2 q, q L^2, q L'^2, C, \Pi, q P^2, q P'^2.$	8 ^e
			$\Lambda^2 q, 2 q L^2, \circ C, 2 q P.$	9 ^e
		d'ordre impair.	$\Lambda^{2q+1}, \circ L^2, \circ C, \circ P.$	10 ^e
			$\Lambda^{2q+1}, \circ L^2, C, \circ P.$	11 ^e
			$\Lambda^{2q+1}, \circ L^2, \circ C, \Pi.$	12 ^e
			$\Lambda^{2q+1}, (2q+1) L^2, \circ C, \circ P.$	13 ^e
			$\Lambda^{2q+1}, \circ L^2, \circ C, (2q+1) P.$	14 ^e
			$\Lambda^{2q+1}, (2q+1) L^2, C, (2q+1) P^2.$	15 ^e
			$\Lambda^{2q+1}, (2q+1) L^2, \circ C, \Pi, (2q+1) P$	16 ^e
	sphéroédrique	quaterternaire..	$4 L^3, 3 L^3, \circ C, \circ P.$	17 ^e
			$4 L^3, 3 L^3, C, 3 P^2.$	18 ^e
			$4 L^3, 3 L^3, \circ C, 6 P.$	19 ^e
			$3 L^4, 4 L^3, 6 L^2, \circ C, \circ P.$	20 ^e
			$3 L^4, 4 L^3, 6 L^2, C, 3 P^4, 6 P^2.$	21 ^e
	décemternaire..		$6 L^5, 10 L^3, 15 L^2, \circ C, \circ P.$	22 ^e
			$6 L^5, 10 L^3, 15 L^2, C, 15 P^2.$	23 ^e
Dans ce tableau, q est un nombre entier quelconque, positif, et au moins égal à 1.				

Table 1: Classification of polyhedra (Bravais 1849)

In the first half of the 19th century, the aim pursued by the scientists who established the basic notions of geometrical symmetry was to obtain a powerful tool in order to progress in the knowledge of the structure of solid matter. After them, some mathematicians would follow the same line as crystallographers in order to develop this new scientific field.

5 THE TWO HUNDRED AND THIRTY SPACE GROUPS OF SYMMETRY

The works of Bravais have inspired numerous scientists. Here are presented four of them, two mathematicians (C. Jordan and A. Schoenflies) and two crystallographers (L. Sohncke and E. Fedorov). This is not an arbitrary choice because all the four of them have produced significant contributions to the field of symmetry.

The interest of mathematicians for the notion of group (elaborated by Galois around 1830) began in the sixties and C. Jordan was one of those who studied and classified groups in order to study this object thoroughly. In 1868 and 1869, he published a paper where he studied the groups' motions which can be finite or infinitely small translations or rotations. Following Bravais, he combined translations and rotations but some motions are not permitted in crystallography. The classification gave one hundred and seventy-four groups. For the first time, symmetry operations were considered and combined. Groups are no more seen as the mere coexistence of symmetry elements. The notion of subgroup appeared under its crystallographic name of hemiedry.

Leonhard Sohncke (1879), a physicist and a crystallographer, took up again this problem of the association of translation groups with orientation groups of polyhedra. He no longer assumed as Delafosse and Bravais, the uniqueness of the orientation in space of molecular polyhedra and he explicitly introduced a mixed symmetry operation, the screw axis which combines a translation and a rotation. Using the method of Jordan, after the rectification of some errors of the paper of this last author, he found out sixty-five crystallographic space groups (sixty-six but two are identical). One can think that for Sohncke, the introduction of improper motions (as reflexion) would involve the existence of two sorts of particles (one "right" and one "left") and that he did not find this situation representative of reality.

The Russian mineralogist E. Fedorov who wanted to resolve the problem of the filling of space by polyhedra followed the way of Sohncke and he demonstrated the existence of two hundred and thirty space groups or possible distributions of identical objects in three dimensional space. A new type of mixed motion was introduced, a translation combined with a reflexion (or glide plane). The publication dates back to 1890.

The mathematician A. Schoenflies, disciple of F. Klein, studied the same problem – classifying the crystallographic groups of motion – along the way outlined by Jordan. Publication of his results (1891) followed by several months that of Fedorov.

CONCLUSION

From the observation of natural crystals with their geometrical regularities as plane faces of polygonal forms, the question encountered by numerous scientists was that of a microscopic hidden order which could explain the obvious external order displayed by crystals. From Kepler to Fedorov, it was the same question.

Throughout this long exploration, physics and chemistry questions were tackled and many of them have been fruitful in these fields. But the pertinent analysis of the geometrical order (observed or theoretical) allowed Haüy and his successors to discover and to understand symmetry, a new type of harmony. Following the meeting of symmetry, the group theory in the second half of the last century, led to an unbelievable scientific fertility.

But even if symmetry was discovered during the nineteenth century, clearly, artists and scientists of antiquity knew something about symmetry as Egyptian pyramids or Platonic solids still show us to this very day.

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