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The Modulor and the Golden Functions

PAPER

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The Modulor is a widely known in the modern architecture theory system of proportions created by the French architect Le Corbusier [1] (see Fig. 1). In spite of the limitations of its practical dissemination it is the result which integrates modern scientific knowledge and reflects objective laws of form building in arts. The key principle of the Modulor system is, undoubtedly, its geometrical idea. Nevertheless, mathematical interpretation of this idea has not been given due in the numerous investigations and papers devoted to the Modulor.

As it is known, the Modulor creator sets the aim to establish the integer system of proportions. But this system is formed by means of the irrational numbers rounding of the so-called golden series

$(\ldots\Phi^{-n},\Phi^{-n+1},\ldots,\Phi^{-1},\Phi^{0},\Phi^{1},\Phi^{2},\ldots\Phi^{n},\ldots,$ where $\Phi=\frac{\sqrt{5}+1}{2}\approx 1{,}618\ldots$–golden proportion). Thus Le Corbusier proceeds from the fact that the integer meanings presented in the Modulor in the form of the so-called red and blue scales, are the expressions of the exponential function. In particular, for the red scale $R_n = k\cdot\Phi^n$, and for the blue scale $S_n = 2k\cdot\Phi^n$, $n=\Lambda\ -2,-1,0,1,2,\Lambda$, k – dimensional coefficient, which approximately equals 1,13. According to this,

$$\frac{R_n}{R_{n-1}}\approx\Phi \text{ and } \frac{S_n}{S_{n-1}}\approx\Phi \qquad (1)$$

Just in this sense the modern investigators of the Modulor understand its mathematical essence [2].

Le Corbusier transfers additive properties of the golden series onto the integer series as well, that is he admits that

$$R_n = R_{n-2}+R_{n-1} \text{ and } S_n = S_{n-2}+S_{n-1}. \qquad (2)$$

But in reality the additive law (2) is not strictly realized here. Its violation is observed on the red scale at numbers 267, 432, 698 and also 698, 1130, 1829, on the blue scale - at numbers 11, 18, 30 and 330, 534, 863.

It is evident that strict reproduction of the golden series properties in the integer scales structure of integer scales is impossible. Solution of the problem set in the Modulor follows from mathematical results of phillotaxis investigation [3]. This investigation revealed the possibility of mathematical interpretation of the golden section and Fibonacci numbers basing on Minkowski geometry.

Let's consider Fig. 2. It shows such "connection" of the square grid with the Minkowski plane coordinate system, under with, in particular, the vertices coordinates X and Y situated on the hyperbolas of a singular radius ($a = 1$) are Fibonacci numbers F. Thus, we obtain the following expressions: $M_1\,[0;1]$, $M_2\,[1;1]$, $M_3\,[1;2]$, $M_4\,[2;3]$, $M_5\,[3;5]$.... Here the hyperbola equation has the form:

$$X^2 \pm XY - Y^2 = \pm a^2. \tag{3}$$

It's also noteworthy that

$$\frac{y_{M_2}}{y_{M_1}} = \frac{y_{M_3}}{y_{M_2}} = \Lambda = \frac{y_{Mn}}{y_{Mn-1}} = \Phi = \frac{x_{M_{n-1}}}{x_{M_n}} \tag{4}$$

That is, the golden series is realized along the Ox (or Oy) axis by means of the coordinates of consecutive vertices lying on symmetrically situated hyperbolas:

$$y_{M_1} : y_{M_2} : y_{M_3} : \ldots : y_{M_n} = \Phi : \Phi^2 : \Phi^3 : \ldots : \Phi^n. \tag{5}$$

This concerns vertices lying on any hyperbola.

If a unit hyperbolic angle n is represented by a semi-angle dividing two consecutive vertices situated on one branch of the hyperbola, and the counting out of the angular coordinate is done according to the position of the axis Ox, the position of a singular hyperbola vertices in the Oxy coordinate system is described by the formulas:

for even numbers n (for vertices belonging to hyperbola $X^2 + XY - Y^2 = 1$):

$$X = \frac{2}{\sqrt{5}} \cdot \frac{\Phi^{n-1} + \Phi^{-n+1}}{2}, \quad Y = \frac{2}{\sqrt{5}} \cdot \frac{\Phi^n - \Phi^{-n}}{2}; \tag{6a}$$

for odd numbers n (for vertices belonging to hyperbola $X^2 - XY - Y^2 = -1$):

$$X = \frac{2}{\sqrt{5}} \cdot \frac{\Phi^n - \Phi^{-n}}{2}, \quad Y = \frac{2}{\sqrt{5}} \cdot \frac{\Phi^{n+1} + \Phi^{-n-1}}{2}. \qquad (6b)$$

We introduce the concept of golden hyperbolic functions (or golden function):

$$\frac{\Phi^n + \Phi^{-n}}{2} = Gch\, n - \text{ golden cosine,}$$

$$\frac{\Phi^n - \Phi^{-n}}{2} = Gsh\, n - \text{ golden sine, etc.} \qquad (7)$$

Now the formulas (6a), (6b) have the form:

for even n: $$X = \frac{2}{\sqrt{5}} \cdot Gch\,(n+1), Y = \frac{2}{\sqrt{5}} \cdot Gsh\, n; \qquad (8a)$$

for odd n: $$X = \frac{2}{\sqrt{5}} \cdot Gsh\, n, \quad Y = \frac{2}{\sqrt{5}} \cdot Gch\,(n+1). \qquad (8b)$$

Let's consider the expressions of ordinates Y of the points $M1, M2, M3, \ldots$:

$$\begin{aligned} Y_{M_1} &= \frac{2}{\sqrt{5}} \cdot Gch\, 1 = 1 = F_1, \\ Y_{M_2} &= \frac{2}{\sqrt{5}} \cdot Gsh\, 2 = 1 = F_2, \\ Y_{M_3} &= \frac{2}{\sqrt{5}} \cdot Gch\, 3 = 2 = F_3, \\ Y_{M_4} &= \frac{2}{\sqrt{5}} \cdot Gsh\, 4 = 3 = F_4, \\ &\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \qquad (9)$$

Generalizing, we have:

$$F_{n-1} = \frac{2}{\sqrt{5}} \cdot Gch\,(n-1),$$
$$F_n = \frac{2}{\sqrt{5}} \cdot Gch\, n, \tag{10}$$

where n is an even number.

In general, the vertices coordinates X and Y belonging to any hyperbola, are always the terms of certain recurrent series, to which the rule (2) is extended. The u_k term formula of such a series represents the hyperbola parameter a and the angular vertex coordinate Which in general case may have the non-integer value:

$$u_n = \alpha \cdot \frac{2}{\sqrt{5}} \cdot Gch\,(n+\Delta),$$
$$u_{n+1} = \alpha \cdot \frac{2}{\sqrt{5}} \cdot Gch\,(n+1+\Delta), \tag{11}$$

where $0 < \Delta < 1$, Δ is the fractional part of the hyperbola angle.

But let's return to the Modulor and compare series R of the red scale with the Fibonacci series:

$$R = \{6, 9, 15, 24, 39, 63, 102, 165, 267, 432, 698, 1130, 1829, \ldots\},$$
$$F = \{2, 3, 5, 8, 13, 21, \ldots\}. \tag{12}$$

As we see, R-succession is the threefold Fibonacci series:

$$R_n = 3 \cdot F_{n-2}. \tag{13}$$

In its turn,

$$S_n = 6 \cdot F_{n-2}. \tag{14}$$

It follows that the terms of R-succession are the coordinates of the square grid vertices belonging to hyperbolas of the radius $a = \pm 3$. By analogy with the numbers of S-succession we obtain $a = \pm 6$.

Thus, the Modulor acquires a strict mathematical interpretation due to its certain attachment to the Minkowski plane coordinate system (see Fig. 3). Now it is evident that the Modulor integer and irrational scales are the expressions of different mathematical regularities – correspondingly, of the golden functions $Gsh\,n$ and $Gch\,n$, and the exponential function Φ^n, so the attempt to match them inevitably leads to inaccuracies which actually were manifested in the Modulor authentic variant.

Our investigation gives every reason to talk about the possibility of development of a new analytical direction in the theory of architectural proportions. In this case the Minkowski geometry and the golden functions appear to be the universal regularities exhibiting fundamental connection of artistic methods with the unique regularities of form building in nature.

References

1. Le Corbusier, Charles. *The Modulor*. Moscow: Strojizdat, 1976 (in Russian).

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3. Bodnar, Oleg Ya. "The Phillotaxis Geometry." *Dopovidi Akademiji Nauk Ukrajiny* , no. 9 (1992), pp. 9-14 (in Ukrainian).