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CIRCLE PROBLEMS ARISING FROM WASAN

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Abstract: *Many wasan problems seem to invite generalisations or related questions; here are a few examples. This article is the result of discussions with Hiroshi Okumura during the ISIS-Symmetry Congress in Washington D.C., USA, in August 1995.*

Problem 1 was found by Hiroshi Okumura in a Wasan book in the Asian Division of the Library of Congress in Washington DC (Okayu 1840).

Problem 1. In Figure 1, *PQRS* is a rectangle, the three circles *A*, *B*, *C* of equal radius touch the various lines as shown, and *B* and *C* touch each other. Show that the lengths *QR* and *TR* are equal.

It turns out that this situation can occur for only one shape of rectangle. (Figure 1 has been drawn inaccurately so as not to reveal this extra property without due investigation.) For this reason I decided to try to generalise the result; I conjectured and then proved the theorem now stated as Problem 2.

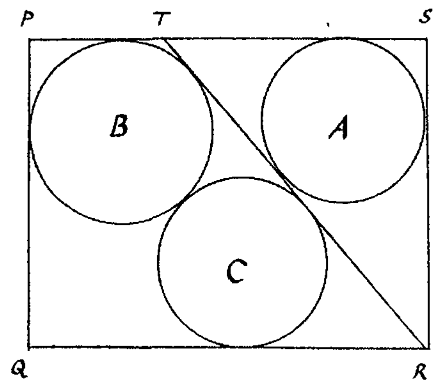

Figure 1

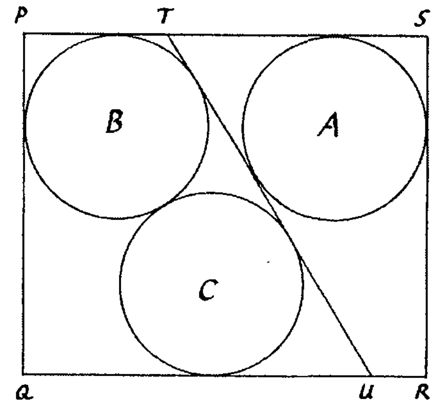

Figure 2

Problem 2. In Figure 2 we have the same situation as in Figure 1 except that the line TU no longer passes through R. Show that the lengths QR and TU are equal.

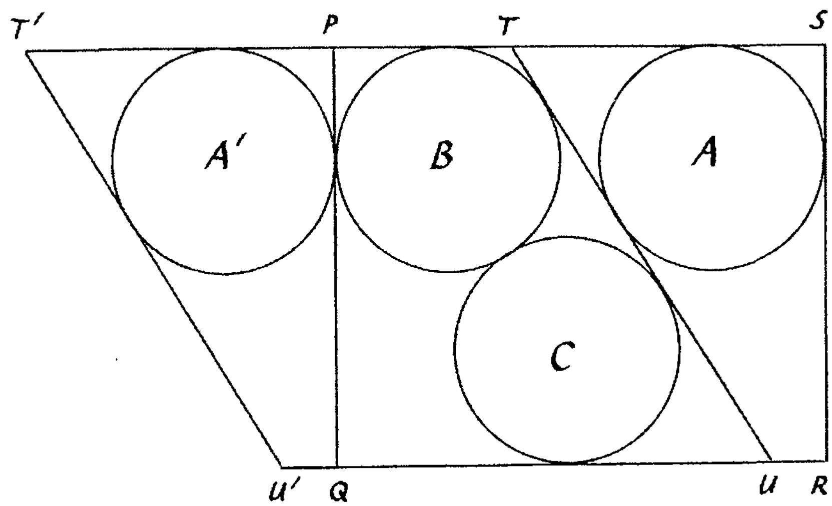

Figure 3

Solution. In Figure 3 the quadrilateral $T'PQU'$ has been drawn congruent to $TSRU$, so A' has the same radius as the other circles, and A' and B touch each other. Then the figure of three touching circles A', B, C has a line of symmetry passing through the centre of B. Hence the parallelogram $T'U'UT$ touching these circles has the same line of symmetry; hence it must be a rhombus and $U'T$ is the line of symmetry. Hence $QR = U'U = TU$.

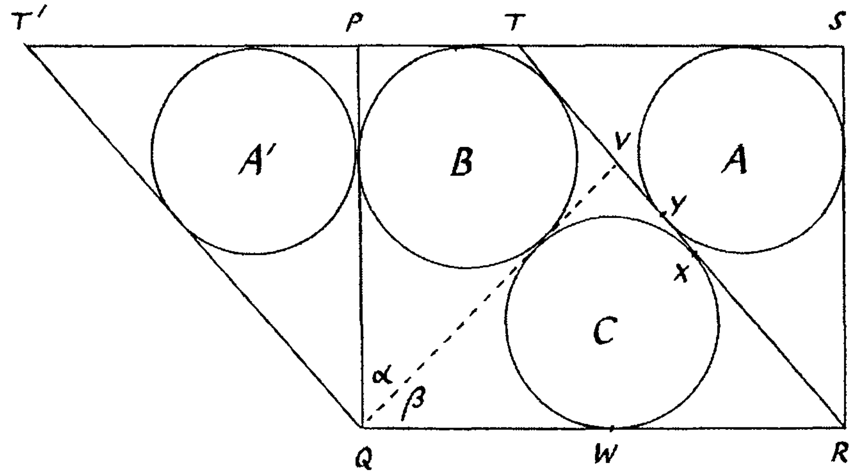

Figure 4

Problem 1 is a special case of Problem 2, but we can investigate Figure 1 further. Applying the method of solution of Problem 2 to Figure 1 we obtain Figure 4. Reflection of the lines QP and TT' in QT shows that the common tangent QV of B and C is perpendicular to TR. Also C is the reflection of B in QV, so the angles α and β are equal; hence $\alpha = \beta = 45°$. Hence the various angles in the figure are all multiples of 45°. Hence $TR = QR = 2WR = 2XR$. Also $TR = 2YR$, so $X = Y$ and the circles A and C touch each other; a result that would have been immediately revealed or conjectured from an accurate drawing of Figure 1. The ratio of the sides of the original rectangle $PQRS$ in Figure 1 is now seen to be $\sqrt{2}$.

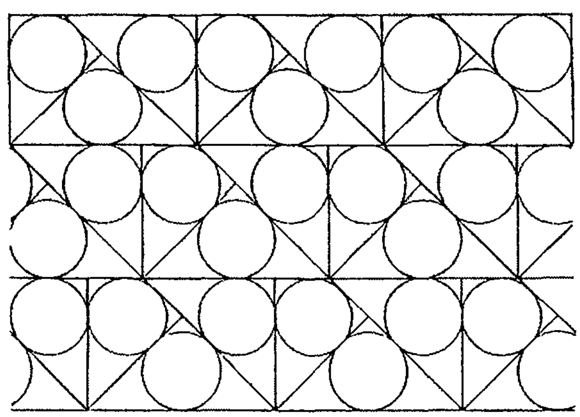

Figure 5

The solution of Problem 2 was essentially obtained by placing next to the original rectangle (with its circles and the oblique line) an identical copy of it, thus revealing an unexpected symmetry. If we lay out rows of these rectangles like bricks, each row being suitably staggered relative to the previous one, we can obtain a pattern covering the plane, of symmetry type *cm* (Grünbaum and Shephard 1987, p. 40). Such a pattern can be obtained from the general rectangle in Figure 2, but it is here illustrated in Figure 5 derived from the special case of Figure 1. It is best thought of as a tiling of rhombi; its symmetry is not easy to appreciate visually because its lines of symmetry make an angle of 22.5° with the vertical.

The next problem, with a long solution, is given by Sakuma (1877), and Okumura has given a much simpler solution (Okumura, 1995).

Problem 3. In Figure 6 the various circles touch each other as shown and the figure has central symmetry about the centre of the outer circle R of radius r, so that the circles A and A' touch R at the ends of a diameter of R and have equal radius a say, B and B' have equal radius b, and C and C' have equal radius c. Also the centres of A, B, C, A', B', C' form a convex hexagon. Show that $r = a + b + c$.

If we replace the circles in Figure 6 by cycles, by indicating an orientation for each circle by means of an arrow as shown, in such a way that the cycles antitouch, we see that R has clockwise orientation so that its radius is regarded as negative. As Okumura points out, the relation between r, a, b, c now becomes $r + a + b + c = 0$. Okumura has also shown that the centres of the seven circles form the vertices of six congruent triangles as shown in Figure 6.

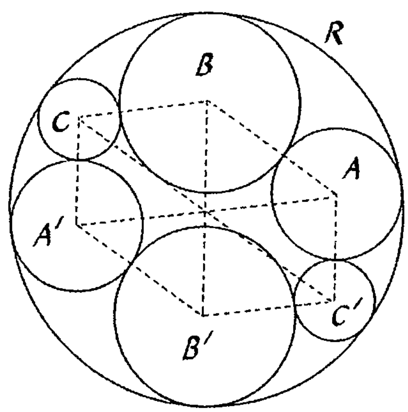

Figure 6

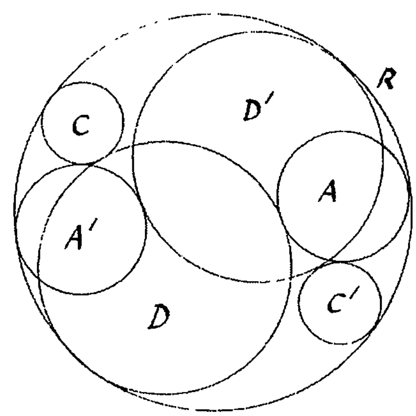

Figure 7

In the situation of Figure 6, there is another cycle D, of radius d say, antitouching R, A and C, and another cycle D', also of radius d, antitouching R, A' and C'. These various cycles are shown in Figure 7. There seems to be nothing special that can be said about their centres, but during a conversation with me Hiroski Okumura considered the question: what is the relation between r, a, c and d? After a tedious calculation using coordinates I found the answer, but in true Wasan fashion I will pose the problem below without giving a solution, in the hope that a shorter one may be found.

Problem 4. In Figure 7 as described above, recalling that r is negative, show that

$$r^3 + (a+c+d)r^2 - 4acd = 0.$$

In Figure 6 and 7 combined, the cycles A, B, C, D form a chain of cycles each antitouching the next and all antitouching R; but A and C have a special relationship: C also antitouches A'. It is natural to ask a more general question: if A, B, C, D are four general cycles, each antitouching the next, D antitouching A, and all antitouching R, as in Figure 8, what is the relation between their radii a, b, c, d and r? The relation looks simpler when expressed in terms of $\rho, \alpha, \beta, \gamma, \delta$, the reciprocals of r, a, b, c, d. Once again my solution, using inversion, is too long to give here, so I merely pose a problem, which has no doubt been solved elsewhere in the literature.

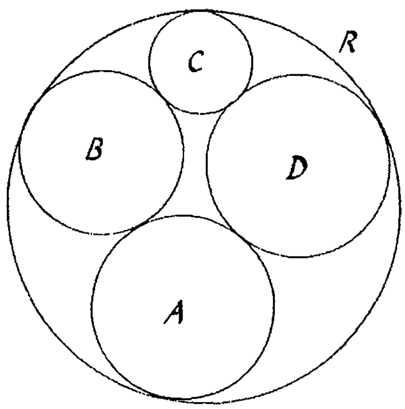

Figure 8

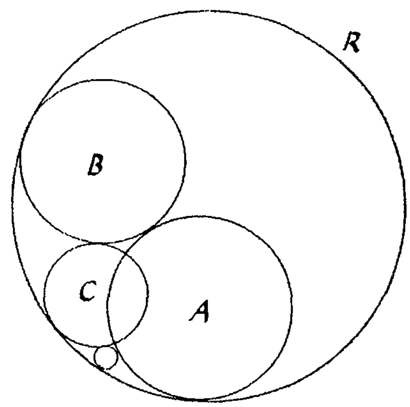

Figure 9

Problem 5. In Figure 8 as described above, show that

$$16\rho^4 - 8[2\alpha\gamma + 2\beta\delta + (\alpha+\gamma)(\beta+\delta)]\rho^2 - 16[\beta\gamma\delta + \alpha\gamma\delta + \alpha\beta\delta + \alpha\beta\gamma]\rho - \\ - 16\alpha\beta\gamma\delta + (\alpha-\gamma)^2(\beta-\delta)^2 = 0.$$

This expression is a quadratic in α, β, γ, and δ, which is what we should expect: if r, a, b, c are given, there are two different ways of positioning the antitangent cycles A, B and C relative to R, as shown in Figures 8 and 9, giving two possibilities for D with different radii.

As partial check of the correctness of the expressions in Problems 4 and 5, my calculations were asymmetrical but the final expressions have the required amount of summetry. Also Figure 10, with cycles of radii 3, 2 and 1 antitouching each other and antitouching the outer cycle of radius –6, provides various special cases of Figures 7 and 8 from which the correctness of the constants in the expressions can be checked.

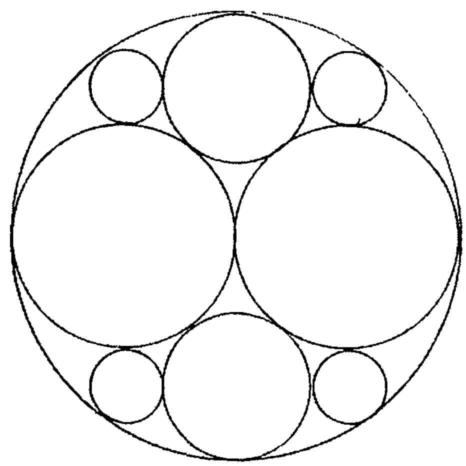

Figure 10

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