Symmetry: Culture and Science
Vol. 8, No. 1, 55-57, 1997

AN INCORRECT SANGAKU CONJECTURE

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Problem 1.2.7 in Fukagawa and Pedoe (1989) reads as follows. *In Figure 1, which is symmetrical about the diameter AB, if r denotes the radius of the small circle, show that*

$$1/r = 1/FE + 1/AF.$$

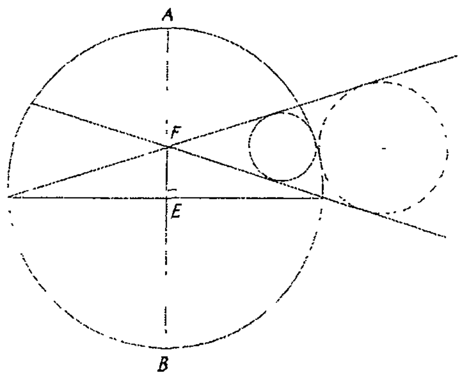

Figure 1

The problem dates from 1878 in a surviving tablet in the Hyogo prefecture, and the result is not difficult to prove using coordinate geometry. It can also be verified that *if* r' *denotes the radius of the broken circle then*

$$1/r' = 1/FE - 1/FB.$$

Hiroshi Okumura has brought to my attention a generalisation of the above problem in *The Sangaku in Gunma* (1987), p. 72. *In Figure 2, if* r *and* s *denote the radii of the two small circles, show that*

$$1/\sqrt{rs} = 1/FE + 1/AF,$$

or equivalently, show that

$$\mathbf{E} = 1,$$

where **E** denotes the expression $\sqrt{rs}\ (1/FE + 1/AF)$.

Any investigation of the lengths occuring in this expression involves awkward calculations, but I eventually began to suspect, for various reasons, that the result is not true. To verify this we need only find a single counterexample, but we shall go further and show that **E** *can take all positive values.*

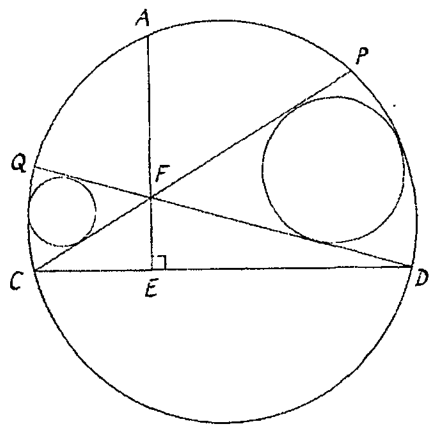

Figure 2

I first considered the case in which CP is a diameter, with DQ perpendicular to it, as in Figure 3. Take DQ and CP as cordinate axes; assume that the large circle has radius 1 and centre O $(0, -\cos 2\theta)$. Calculations using coordinate geometry and trigonometry show that the circles α and β have radii $r = 2(\sin\theta - \sin^2\theta) = 2\sin\theta(1 - \sin\theta)$ and $s = 2(\cos\theta - \cos^2\theta) = 4\cos\theta\sin^2(\theta/2)$; also $FE = \sin 2\theta\cos\theta = 2\sin\theta\cos^2\theta$. The line AB has equation $y = -x\tan\theta$, and if we investigate where this line meets the circle $x^2 + (y + \cos 2\theta)^2 = 1$,

$$AF = \sin\theta(\sqrt{4\cos^4\theta + 1} - \cos 2\theta).$$

Consider now what happens as θ varies between $\pi/4$ and 0. We see that **E** can be written in the form $\mathbf{E} = \sin(\theta/2)f(\theta)/\sqrt{\sin\theta}$, where $f(\theta)$ is a continuous positive function of θ that tends to a finite limit as $\theta \to 0$. Hence $\mathbf{E} \to 0$ as $\theta \to 0$. But we know that

$$\mathbf{E} = 1$$

in the symmetrical case when $\theta = \pi/4$. Hence **E** takes all positive values ≤ 1 as θ varies between $\pi/4$ and 0.

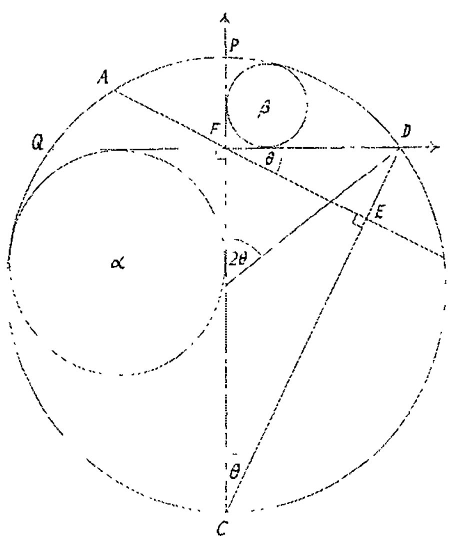

Figure 3

A simpler special case is shown in Figure 4, in which CQ and XY are perpendicular diameters. Suppose the radius of the large circle is 1. Then

$$r(1 + \sec\phi) = XK + KF = 1 + \tan\phi\text{, and } s(1 + \sec\phi) = 1 - \tan\phi.$$

Hence $\sqrt{rs}\sin^2\phi = \cos\phi\sqrt{\cos 2\phi}(\sec\phi - 1)$.

Also $AF = \sec\phi$ and $FE = 2\cos\phi - \sec\phi = \sec\phi\cos 2\phi$.

Hence $(1/AF + 1/FE) = 2\cos^3\phi/\cos 2\phi$ and $\mathbf{E} = 2\cos^4\phi(\sec\phi - 1)/\sin^2\phi\sqrt{\cos 2\phi}$.

Hence $\mathbf{E} \to \infty$ as $\phi \to \pi/4$. Also $\mathbf{E}$ is positive and continuous, and $\mathbf{E} = 1$ when $\phi = 0$. Hence $\mathbf{E}$ takes all positive values ≥ 1 as ϕ varies between 0 and $\pi/4$.

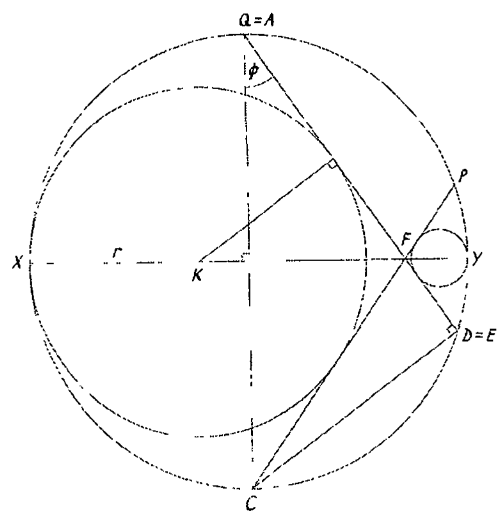

Figure 4

When $\theta = 26°$ as in Figure 3, we find that $\mathbf{E}$ = .954..., and when $\phi = 35°$ as in Figure 4, $\mathbf{E}$ = 1.033... . Hence $\mathbf{E}$ can be close to 1 even in figures that are far from being symmetrical, so it is not surprising that the original conjecture was made, probably as the result of careful drawings.

REFERENCES

Fukagawa, H. and Pedoe, D. (1989) *Japanese Temple Geometry Problems*, Winnipeg, Canada: The Charles Babbage Research Centre.

The Sangaku in Gunma (1987) Gunma Wasan Study Association, (in Japanese).