Symmetry: Culture and Science
Vol. 7, No. 4, 1996, 413-441

SYMMETROSPECTIVE

GOLDEN SECTION(ISM): FROM MATHEMATICS TO THE THEORY OF ART AND MUSICOLOGY[1]
PART 1

In memoriam Ernő Lendvai (1925-1993)

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Abstract: *The golden section,* a/b = b/(a+b), *can be considered in both space and time (Chap. 1). We survey this concept in mathematics and art, including the terms "extreme and mean ratio" (Greek mathematics), "proportion of a middle and two ends" (Islamic mathematics), "divine proportion" (Pacioli, 15th c.), "divine section" (Kepler, 17th c.), "golden section" (19th c.), and some widespread misunderstandings in the literature (Chap. 2). Among others, we demonstrate that the surviving documents on the proportional systems of Polykleitos, Vitruvius, and Dürer, respectively, cannot refer to the golden section, despite the fact that very many authors claim that this concept is clearly used by them. We present our new interpretation of the Leonardo-Pacioli cooperation with the conclusion that Leonardo's "divine*

[1] Our expression *golden sectionism*, i.e., considering the golden section as a distinctive doctrine (which is a usual definition of an *-ism*), is formed on the analogy with the existing term *golden numberism*. The latter was introduced by Roger Herz-Fischler, or Roger Fischler in works written before 1982, a Canadian mathematician, who published many works on the golden section (cf., the chapter "Golden numberism" in his paper in the *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, Vol. 2, London, 1994, pp. 1580-1583).

In the present paper we prefer "golden sectionism", i.e., we give more emphasis to the geometrical aspects against the numerical ones. It is clearly broader than the "cult of the golden section", the expression that was used by Christopher Alexander, a specialist of theory of architecture ("Perception and modular co-ordination", *Journal of the Royal Institute of British Architects*, 66 (1959), p. 425). Although golden sectionism (or golden numberism) is not a fortunate expression from stylistic point of view, we have no better term to describe a movement that will be discussed in this paper (see in more detail Chap. 3.1). Incidentally, there is a point where we should correct Herz-Fischler. In the cited paper (p. 1582) he claims that golden numberism was limited to Germany until the early part of the 20th century. Herz-Fischler obviously missed some Russian, Hungarian, as well as Japanese works (see Chaps. 3.2 and 3.3).

proportion" is a metaphorical expression without a specific meaning, which was later adopted as a mathematical term for the golden section by Pacioli. (This encompasses a new dating of a section of Leonardo's Trattato della pittura.*) We also study the formation of the first terms that refer to gold,* der goldene Schnitt *and its Latinized version the* sectio aurea *in the German mathematical-educational literature of the 19th century. While many historians of science and linguists refer to M. Ohm's book (1835) as the first known appearance of the German term in a printed work, we give an earlier example in F. Wolff's textbook (1833). After this, we discuss how 'golden sectionism' appeared in the German and Russian theories of art following Zeising's books in the 1850s and its Russian summary in the 1870s (Chap. 3). This led to various claims that the golden section was used extensively in art works of the past. On the other hand, we reject the possibility of 'proving' the usage of the golden section by simple measurements. We present clear guide-lines for detecting the golden section in art. We also discuss golden sectionism in Japan which was partly a Western influence, partly a topic in traditional mathematics (Yamagami, 1843). Following this, we investigate how the concept of the golden section reached the field of musicology in the German speaking countries (Zeising, 1854; Liszt, 1859), in Russia (Vipper, 1876; Rozenov, 1904; Sabaneyev, 1925-27), and then, independently of them, in Lendvai's systematic study of Bartók's music (from the late 1940s) and more recently Howat's book (1983) on Debussy (Chap. 4). Finally, we discuss the future of the studies related to the golden section. It is quite interesting that the 'application' of the golden section is widely known in such cases where it was not used at all, while some actual documents on the golden section (e.g., Liszt's letter of 1859) are almost unknown. We need a balanced view on the golden section which approaches to a 'mean' between the 'two extremes' (Chap. 5).*

1 THE GOLDEN SECTION: NOT ONLY IN SPACE, BUT ALSO IN TIME

The *golden section*, although under other names, is well known in geometry since ancient times, and later it also gained some importance in art as a proportion that symbolizes the harmony between the parts and the whole, and which is associated with organic shapes, including the human body. The term *golden section* (golden cut, division, mean, measure, proportion, or ratio) refers to the division of a line-segment, which could also be a time-interval, into two parts so that the ratio of the smaller part to the larger one is the same as the ratio of the larger one to the whole, i.e.,

$$a/b = b/(a+b)$$

Note that in the literature on art, *proportion* (in Greek *analogia*, from *ana* + *logos*) became confused with *ratio* (in Greek *logos*) since the Roman period when these two terms were translated into Latin as *proportio* (*pro* + *portio*) and *ratio.*

Eventually, proportion became a general expression that refers to a ratio of two magnitudes (a/b) or a set of ratios (a/b, c/d, ...) where the pairs of magnitudes are associated with the parts and the whole of an object. The concept of proportion has a great importance in aesthetics since it can be quantified and used for comparison. However, the original meaning of proportion, which we will use in this paper, is not just a ratio or a set of ratios, but the equality of two (or more) ratios. It can be described by the following equation:

$$a/b = c/d$$

Here we have four variables and any three of them determine the fourth one. This equation has a special case that is frequently mentioned in both mathematical and musicological works since ancient times, the definition of the *geometric mean* (b) between two given magnitudes, lengths, or numbers (a and c):

$$a/b = b/c$$

It is also referred to as "continuous proportion" because of the repeated b. Thus, we have only three variables and any two of them determine the third one. The golden section can be considered as a special case of the geometric mean where

$$c = a + b$$

Indeed, after this substitution, we have the equation that defines the golden section:

$$a/b = b/c \text{ and } c = a + b \implies a/b = b/(a+b)$$

In this sense the golden section is very simple since its definition uses just two variables a and b, where one of them determines the other one. However, the numerical value of this proportion, the *golden number* (ϕ), is not so simple:

$$\phi = a/b = b/(a + b) = (-1+\sqrt{5})/2 = 0.618...$$

or, if we use its reciprocal, which is distinguished here by an asterisk:

$$\phi^* = b/a = (a + b)/b = (1+\sqrt{5})/2 = 1.618...$$

Because of the presence of $\sqrt{5}$, the golden number is irrational, which cannot be expressed as the ratio of two integers. In other words, the golden section is an incommensurable proportion.

Consequently, one never can use the exact golden section of time-intervals in those musical scores that are based on equal units (e.g., bars): here all marked time-intervals (length of themes, duration of special instructions, etc.) can be measured by integer times of the corresponding unit, therefore all the possible ratios are rational. Still, in experimental music there are no such limits: we mention here just one example that is based on an impressive way of transforming the golden section from space to time, specifically from line-segments into time-intervals. In the case of tape music, which is partly recorded in advance and then performed in live concerts with the participation of musicians, one could easily measure the length of the used tape, thinking especially about its old form before the spread of cassettes, and mark the golden section point as the division of themes. Of course a measurement is never exact, but the composer may mark this point as exactly as he wishes beyond the scope of the available methods. (The advantage of measuring tape-intervals by ruler instead of time-intervals by clock is the fact that we have better orientation in the visual space where the processes, including the motions of eye, are reversible.) This idea was used by the contemporary Greek-French composer Iannis Xenakis. Earlier a similar method was pioneered in the art of moving pictures by the Russian director Sergei Eisenstein (Eizenshtein) who considered a set of golden section points, as the beginnings of important scenes, by measuring the celluloid film in the case of his movie *Bronenosets Potemkin* (*The Battleship Potemkin*, 1925). Such precise measurements, which are the 'secrets' of the composer or movie director and hardly can be detected by the audience, are very rare. However, there is a natural way to indicate one's commitment to the golden section in time, including classical music, by using ratios of Fibonacci numbers.

The Italian mathematician Fibonacci (early 13th c.) described a sequence of numbers that starts with two 1's and then each further term is the sum of the previous two ones: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ... There are conjectures that this sequence was known in various cultures much earlier, but, in fact, it became widely known following Fibonacci's book. He presented it in the framework of a very artificial exercise about the growth of the number of rabbits, without giving any further detail in his book. Later it was discovered that the ratios of the neighboring Fibonacci numbers tend to the golden number:

1/1 = 1; 1/2 = 0.5; 2/3 = 0.666...; 3/5 = 0.6; 5/8 = 0.625;
8/13 = 0.615...; 13/21 = 0.619...; 21/34 = 0.617...; 34/55 = 0.618...; etc.

The first known document indicating this property is a hand-written annotation in an early 16th century printed copy of Euclid's *Elements*; this document was found by Curchin and Herz-Fischler more recently (*Centaurus*, 28 (1985), pp. 129-138). Earlier this discovery was frequently attributed to the German astronomer Kepler who referred to it in an 1608 letter. Note that the ratios of neighboring terms also tend to the golden number in the case of the broader set of Fibonacci-type sequences: here the first two numbers are arbitrary integers (e.g., 1 and 3), while the further terms are defined by the same additive rule (1, 3, 4, 7, 11, 18, etc. - this special case is often called the Lucas sequence). In the 1830s the German and French scholars A. Braun, C. F. Schimper, L. and A. Bravais discovered a strong tendency of the presence of neighboring Fibonacci numbers at the arrangement of leaves (phyllotaxis) on a stem and the numbers of left and right spirals on the

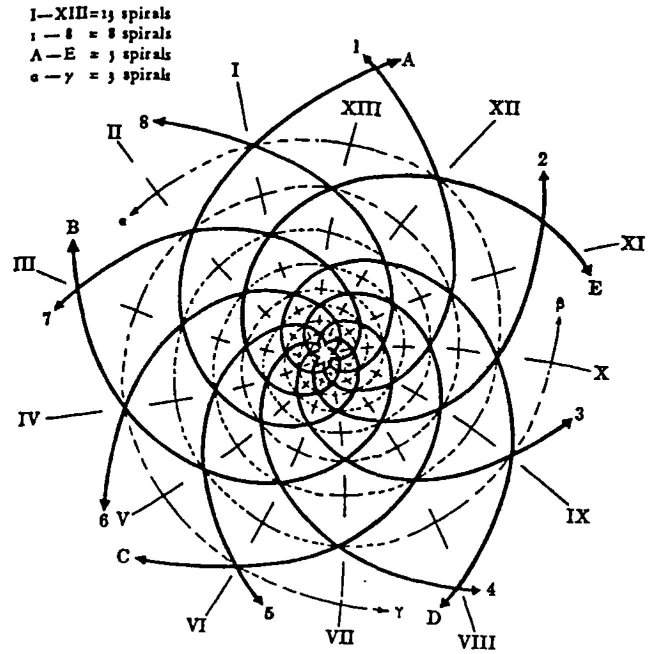

Figure 1: Spirals on the surface of a pine-cone: 3, 5, 8, and 13 spirals. The musicologist Ernő Lendvai's analysis in his book *Bartók dramaturgiája* (Bartók's Dramaturgy, in Hungarian, Budapest, 1964, p. 177; also see his *Symmetries of Music*, Kecskemét, 1993, p. 21).

surface of a pine-cone, a pineapple, a sunflower, etc. (Fig. 1). Some spirals in nature were compared with the golden spiral: the logarithmic spiral that intersects any straight line through its center according to a sequence of golden section points. These mathematical and biological discoveries gave a new dimension to the topic of the golden section and the related 'approximate proportions' based on Fibonacci numbers. (We cannot speak here, e.g., in the case of $3/5 \approx 5/8$, about exact proportions: the two ratios are not the same, just approximately equal.) The additive property of the Fibonacci numbers (i.e., $1+1 = 2$, $1+2 = 3$, $2+3 = 5$, $3+5 = 8$, $5+8 = 13$, etc.) makes possible, for example, the following division of a line-segment or a time-interval by approximate golden section points (the ratios 1/2, 2/3, 3/5, 5/8, 8/13 successively approximate the golden number):

```
I-/- -I- -/- - -I- - - - -/- - - - - - - - -I
 1/ 2   2 /  3
I- 3 -/--- 5 ---I--- 5 ---/------ 8 ------I
I----- 8 -------/---------- 13 -----------I
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Indeed, this method is useful for dividing musical pieces into various sections, moreover this process may be associated with organic growth and symbolism of natural shapes. It is true that the observation of such approximate proportions in time is still not an easy task, but considering musicians and devoted listeners who follow bar numbers, this is not impossible.

We also have here a brief note to sceptical readers who would immediately argue that the perception of time is subjective, or that the listener cannot 'hear' the golden section, moreover cannot consider any proportion without 'seeing' the whole interval. Indeed, this is a very difficult task, but some listeners may have a talent to 'feel' such relationships. In an earlier paper in German I initiated experimental-psychological research on the possible importance of the golden section in time and detected some preference for this proportion among students of video art (*Polyaisthesis*, 4 (1989), pp. 88-102). Without going into details, we will emphasize a different aspect here. Independently of the fact that one can 'feel' or not the proportional relationships, it is important to know something about the deliberately used (or intuitively preferred) 'secrets' of the composer, to learn such ideas that may help both the art of the performer and the understanding of the listener, and that may give some hint to other composers. The golden section was used as a geometric tool by some visual artists (much less times than 'proportion hunters' claim and without having a definite conclusion of its real aesthetical importance), and some composers may feel that they also need some similar scientific help. They may prefer the mathematical properties of the golden section and the Fibonacci numbers (many composers, from the Pythagoreans to Alban Berg, were obsessed with numerological considerations), or may use these as a symbolic representation of organic growth. Just one example: the Russian-born American composer and educator Joseph Schillinger developed a methodology of composition with many mathematical ideas, including the Fibonacci numbers, and - although there is a controversy about his approach - he helped George Gershwin to 'regain' his creativity during the period when he composed *Porgy and Bess*.

2 A HISTORIC SURVEY OF THE GOLDEN SECTION AND ITS TERMINOLOGY IN MATHEMATICS AND ART UNTIL THE 19TH CENTURY (THE PRELIMINARIES OF GOLDEN SECTIONISM AND SOME WIDESPREAD MISUNDERSTANDINGS)

2.1 The Greek *akros kai mesos logos* (extreme and mean ratio) in ancient mathematics and the lack of evidence for its presence in art

Unfortunately the literature on the usage of the golden section in art is full of statements which are contemporary theories projected back to previous eras without mentioning this aspect. Thus, the 'facts' that the Egyptian pyramids and the Greek temples and sculptures were designed according to the golden section have no clear documentary evidence: the corresponding statements are based on modern measurements and related hypotheses (19th-20th cc.). Obviously, there is a methodological problem here:

- the current dimensions of an ancient object are not identical with the original ones,

- any measurement is approximate and the rounded data may indicate various slightly different possibilities, not just one (we will return to these questions in Chap. 3.4).

Let us start with the Egyptian pyramids. There are at least nine different theories in the literature on the geometric shape of the Great Pyramid of Pharaoh Khufu (or Cheops, around 2,600 B.C.). Two of them are based on the golden section:

The *angle of the slope* of a triangular face

$\alpha = \text{arc}\cos\phi \approx 51.827°$ (cf., Röber, 1855; Jarolimek, 1890)

$\alpha = \text{arc}\tan 2\phi \approx 51.027°$ (cf., Choisy, 1899)

where $\phi = (-1+\sqrt{5})/2 = 0.618...$ is the golden number.

Although the first one approximates very well to the measured angle (50'40" ± 1'5" ≈ 51.844°), the only evidence behind this theory is the (mis)interpretation of a remark by the Greek historian Herodotos (Herodotus, 5th c. B.C.). On the other hand, there is a simpler theory based on a rational number

$\alpha = \text{arc}\tan 28/22 \approx 51.843°$ (cf., Petrie, 1883 and 1892)

that gives an even better agreement with the measured data and is supported by both textual and archaeological evidence, specifically by a mathematical problem in the Rhind Papyrus and a tomb with similar regulating lines, as it was discussed in detail by R. Fischler ("Théories mathématiques de la Grande Pyramide", *Crux*

Mathematicorum, 4 (1978), pp. 122-129; "What did Herodotus really say? Or how to build (a theory of) the Great Pyramid", *Environment and Planning*, B6 (1979), pp. 89-93).

Turning to the Greek and Roman periods, we still cannot report any conclusive evidence for the actual application of the golden section in art. First of all, Vitruvius's *De architectura* (On Architecture, in Latin, 1st c. B.C.), the only surviving ancient book on this field, does not provide any such evidence. Although modern researchers often associate Vitruvius's theory of proportion with the golden section, he did not introduce this concept. We may speak about a loose connection only between his statements on proportions and the main idea of the golden section, specifically the harmony between the parts and the whole of an object. It is true that Vitruvius suggests applying human proportions in architecture, but this is not a commitment to the golden section on his part. He emphasizes that both human and architectural proportions depend on *symmetria* (*syn* + *metron*), which refers to commensurability in ancient mathematics. He also uses the more specific term *commensus proportionis* (commensurable proportions, see, e.g., Book 3, Chap. 1, Para. 2). From a mathematical point of view this is clear evidence against any preference for the golden section, which is an incommensurable proportion. We should admit that the latter argument has a weak point: Vitruvius was not a mathematician and he could have considered the concepts of *symmetria* and *commensus* in a metaphorical sense. However, the numerical data in the text do not support this possibility: Vitruvius favors simple ratios of small integers throughout the work. The only circumscribed incommensurable ratio in the text is $1/\sqrt{2}$. Although he deals with various figures there is no reference to any geometrical construction that may lead to the golden section. Interestingly, he never brings up the importance of regular pentagons. In addition to this, Vitruvius's rich survey of ancient terminology, often citing the original Greek expressions in his Latin text, does not have any expression for the golden section. It is also striking that neither Plato, nor Aristotle refers to the aesthetical aspects of the golden section. Both of them discussed mathematical and aesthetical questions alike and used the following terminology:

- *symmetria* (vs. *asymmetria*) as commensurability in mathematics,
- *symmetria* (vs. *asymmetria*) as due proportion, an aspect of beauty.

Of course we should not stop at Vitruvius, as well as Plato and Aristotle's aesthetical texts: let us see some other ancient fragments on canons and proportional systems. The frequently cited book *Canon* by the sculptor Polykleitos (or Polyclitus, 5th c. B.C.) does not survive with the exception of two fragments. These are quotations in books by Philo and Plutarch, respectively. There are two or three more fragments that we may accept as reliable sources: two by Galen and another one by Plutarch (cf., A. F. Stewart, "The canon of Polykleitos: A question of evidence", *Journal of Hellenic Studies*, 98 (1978), pp. 122-131). On the basis of these, we cannot reconstruct the ratios of this canon beyond the fact that these were based on proper *symmetria* (i.e., commensurability or due proportion) of the parts of the body such as:

$$a/b,\ b/c,\ c/d,\ d/e, \text{ etc.}$$

where a, b, c, d, and e refers, respectively, to the lengths of a finger (or phalange?) (a), another finger (b), the metacarpus and carpus (i.e., the palm and the base of the hand) (c), the forearm (d), and the upper arm (e). Although the fragment emphasizes that "everything to everything else" should form ratios, no further details are specified (Galen's *De placitis Hippocratis et Platonis*, Book 5, p. 449, Kühn edition). Since the Greek term *symmetria* has two meanings, let us consider both possibilities:

(1) *symmetria* means mathematical commensurability.

In this case the golden section is automatically excluded because it is an incommensurable proportion.

(2) *symmetria* means due proportion in art, which is independent of the mathematical meaning.

One may interpret this case that the given ratios should form a system of proportion (equality of ratios):

$$a/b = b/c = c/\mathrm{d} = d/e = \ldots$$

In other words, we have a system of geometric means: b is the geometric mean of a and c; c is the geometric mean of b and d, etc. (the sequence a, b, c, d, ... is a geometric progression). However, this is still not the golden section, which is a special case of the geometric mean ($a/b = b/c$) with a further requirement: $c = a+b$. Likewise, we should require that $d = b+c$, $e = c+d$, etc. Could Polykleitos have had this idea in mind? My answer is: definitely not. A finger plus the palm and the base of the hand, i.e., the full hand, is much shorter than the forearm ($b+c < d$), while the palm and the base of the hand plus the forearm is longer than the upper arm ($c+d > e$). Thus, we have a clear conclusion: although the cited description of the canon is associated with some ratios of the parts, there is no way to connect it with the definition of the golden section. However, some modern researchers have suggested a different approach. Since the available fragments are obscure and do not specify the numerical value of ratios, they have tried to fill the gaps by measuring data. We pointed out the problems of this method at the beginning of this chapter, we will look at further difficulties here. First of all, we have no clear information on the typical measuring-points of ancient sculptors. It is tempting to use the conventions of modern artistic anatomy, but old masters may have had very different methods. There is, however, an even bigger problem. No original sculptures by Polykleitos survive except for some copies made in later periods. The latter statement is also true for Pheidias (or Phidias, 5th c. B.C.), another leading sculptor. In his case we have even less literary data on his preferred proportions. Thus, the frequent statements in modern literature that both Polykleitos and Pheidias intensively used the golden section are mere speculations. Obviously, the measured data of sculptures that are not original cannot give any relevant support to such speculations. Finally, we should also discuss a piece of 'archaeological evidence' in connection with the golden section: an instrument from the Roman period (1st c. A.D.) which is kept in the National Museum of Archaeology in Naples (Museo Archeologico Nazionale, Napoli). Indeed, some experts interpret it as a proportional compass associated with the golden section, but, again, this is a modern view based on measurements.

Are there any records on the preference for ratios of neighboring Fibonacci numbers or other approximate values of the golden number in ancient times? We should keep in mind that neither the clear definition of the Fibonacci numbers (1, 1, 2, 3, 5, 8, 13, ...) nor their connection to the golden section is available in known records before the 13th and 16th centuries, respectively. Still, the additive definition of the Fibonacci numbers is so simple that some people could have used them much earlier. Furthermore, there is a mathematical document, the calculation of the area of a regular pentagon by Hero of Alexandria (*Metrica*, Book 1, Paras. 17-18; 1st c. A.D.), where we may suspect that the golden number was approximated by the ratio of 5/8 (cf., Curchin and Fischler's paper, *Phoenix*, 35 (1981), pp. 129-133). In some sense this is not a very surprising move: in the case of similar problems we also replace the irrational numbers by approximate decimal fractions using a pocket calculator. Going beyond this example by Hero, we may also consider a much broader set of documents where ratios of neighboring Fibonacci numbers are used by chance or due to unknown reasons. It is possible that artists picked up such ratios from the mathematicians and used these without thinking about the sophisticated mathematical methods. Then the usage of such ratios became a rule of thumb (or, if someone wishes, a "golden rule").

Although we cannot claim that these steps represent a conscious manifestation of the golden section in ancient times, they could have contributed to a later aesthetical preference for this ratio. However, for different groups of Greek artists and architects this was not the golden section (singular), but some different ratios (plural!). Here we should refer again to Vitruvius who suggests that the ratio of the breadth to the length of an atrium should be 3/5 (= 0.6), 2/3 (= 0.666...), or $1/\sqrt{2}$ (= 0.707...; see Book 6, Chap. 3, Para. 3). Obviously, this list cannot be interpreted as a systematic approximation of the golden number (0.618...) by Fibonacci numbers: the last ratio is of a different type and relatively far from it. Note that the first two ratios were also considered in the Pythagorean theory of harmony. Consequently, the importance of the ratios 2/3 and 3/5 could have inspired later generations of artists, including some leading figures of the Renaissance, via two different ways:

- either directly, by reading the Vitruvian text,

- or indirectly, by considering musical analogies (the latter is discussed in detail by R. Wittkower in his book *Architectural Principles in the Age of Humanism*, New York, 1971, Part 4).

Are there ratios of larger Fibonacci numbers in the Vitruvian text? In the case of catapults, he uses various unclear symbols for fractions and one of them is usually interpreted as 5/8 (Book 10, Chap. 10, Paras. 4-5; note that Schramm's interpretation is not followed by Granger's modern English translation that gives 2/3). It is not easy to explain the preference for 5/8, but we may mention, beyond the visual intuition, various possibilities, including a more sophisticated musical-mathematical experience (3/5 is the major sixth, but 5/8 can be considered as the minor sixth) or Hero's approximation of the irrational value of the golden number. We should also refer to another widely reported case, the presence of ratios of Fibonacci numbers in the structural patterns of Vergil's poetry. This observation is based on a modern analysis by G. Duckworth and not on statements by Vergil (*Structural Patterns and Proportions in Vergil's Aeneid*, Ann Arbor, 1962). However, a new statistical analysis of Duckworth's data by R. Fischler, whose research field is mathematical statistics, does not confirm the significance of these ratios ("How to find the 'golden number' without really trying", *Fibonacci Quarterly*, 19 (1981), pp. 406-409). As a conclusion, we propose a balanced overview of the importance of ratios of neighboring Fibonacci numbers in ancient times: this is neither a strict refutation, nor a strong support. From a statistical point of view, we cannot speak about a significant preference for such ratios. On the other hand, we should give extra weight to the fact that Vitruvius used the ratios 2/3 and 3/5 in a very important chapter and his work had great influence in later periods. This situation, if it was the actual case, gives a possibility for some modern researchers to ignore the variety among the competing groups of artists and declare that the golden section is the only and ultimate key to the beauty of ancient design. They could be right that there was some "glitter", not definitely consciously, around the golden section (thinking about approximate fractions around the golden number), but they are wrong to over-emphasize its uniqueness.

Finally, let us see the terminological aspects of the topic. The only known ancient expression that clearly refers to the golden section is the Greek *akros kai mesos logos* (extreme and mean ratio), which seems like a periphrasis, not yet a technical term. Considering the definition of the golden section as the equality of two ratios

$$a/b = b/(a+b)$$

a and $a+b$ are the "extremes" and b is the "mean". All of the surviving usages of the expression *akros kai mesos logos* are in mathematical works and their number is surprisingly small (Euclid, 4th-3rd cc. B.C.; Hypsikles or Hypsicles, 2nd c. B.C.; Ptolemy, 2nd c. A.D.; Pappos or Pappus, 4th c. A.D.; Proklos or Proclus, 5th c.

A.D., and an anonymous commentary on Euclid's book). The fact that there is no expression for the golden section in the large body of aesthetical and philosophical works is striking. Thus, there is no direct reference to the golden section in Plato's surviving texts (5th-4th cc. B.C.), although his description of the geometric mean ($a/b = b/c$) is often confused with the extreme and mean ratio ($a/b = b/(a+b)$, i.e., that special case of the geometric mean where $c = a+b$). Still, the legend that Plato had an interest in the mathematical properties of the extreme and mean ratio has an obscure basis. Proklos (5th c. A.D.), who wrote a commentary to Euclid's book, referred to Plato's work concerning the *tomê* (section, cut), which is interpreted by some modern historians of mathematics as an alternative term for the extreme and mean ratio (cf., Euclid's expression *[eis] akron kai meson logon temnein*, cut in the extreme and mean ratio). However, the majority of experts do not support this view and provide different interpretations (e.g., Proklos referred to the sectioning of solids by planes). Even without a definite solution of this problem, we may point to a topic in geometry where the importance of the extreme and mean ratio has been obvious since the era of the Pythagoreans (6th-5th cc. B.C.). This is the theory of those regular polygons and polyhedra that have fivefold symmetry. Since the diagonals of a regular pentagon intersect each other according to the extreme and mean ratio, some ancient (and later medieval) texts on regular polygons and polyhedra used the mathematical properties of this proportion directly or indirectly in calculations.

2.2 The Arabic *nisbat dhalika wasat wa tarafayn* (proportion of a middle and two ends) in the Middle Ages: did Islamic mathematics influence art?

Before turning to the problem of the extreme and mean ratio, it is useful to survey the cultural backgrounds of this period. Although the Greek mathematical traditions declined at the end of ancient times, this field gained a new emphasis in the Islamic world. What was the reason of this specific interest? Why did the Islamic culture prefer Greek works on mathematics, astronomy, and natural philosophy, including Euclid, Ptolemy, and Aristotle, and ignore many other parts of ancient culture? This is obviously a complex problem, but there are some obvious reasons behind this preference:

- the religious sphere required the exact determination of the time of sunrise and of the direction to Mecca from each place, which were complicated mathematical-astronomical problems,

- the trade and navigation in the growing Islamic Empire required more practical mathematics.

This new situation more or less also determined the "position" of the extreme and mean ratio or, as it was called in Arabic mathematical texts, "the proportion of a middle and two ends" (or the proportion of a mean and two extremes). The translations of and the commentaries on Euclid's and Ptolemy's books and a few other Greek works where the extreme and mean ratio is used helped the transmission of this concept, but the practical-mathematical questions did not provide many new challenges to this topic, with the possible exception of some trigonometrical problems in the later periods. (We refer here to the calculation of trigonometric functions in the case of angles that are associated with the regular pentagon and decagon.) However, there is a possible link that we should consider: the relationships between mathematics and art. Geometrical decorations have great traditions in the Middle East since ancient times. Thus, the invention of cylindrically-shaped seals (ca. 3500 B.C.), which were rolled out in clay, led to the mass production of patterns with repetitive motifs. Nomadic tribes developed great skills in carpet-weaving and later these geometric patterns also appeared in the decoration of buildings. In addition to this, the geometric motifs of Arabic calligraphy gave a further inspiration to artisans. However, these traditions alone do not explain the developments in art from the 7th century. The decisive factor was the rise of Islam that prohibited the use of figural representation in religious buildings and contributed to an unparalleled richness of geometric figures in architecture and ornamental art. (Note that the frequent statement that the figural representation is totally missing in the Islamic art is incorrect as the tradition of illustrated manuscripts shows.) Let us focus on the geometric interest of Muslim artisans (craftsmen, artists, masons, architects). Although it is widely believed that, similar to the Hellenistic age, there was no significant cooperation between mathematicians and artisans, this view is not tenable. Various sources indicate that members of the two groups occasionally discussed mutually interesting topics, including the design of

- buildings and some other objects,
- ornamental decorations.

Unfortunately, there are very few clues on architectural design and the used proportions in the surviving documents. The situation is slightly better in the field of ornamental art. Furthermore, the 'deep structure' of the actual design may give some insight into the intuitively used or consciously preferred geometrical principles. We shall study the ornamental art in more details.

The decoration of large surfaces of walls and floors provided a great demand for periodic patterns with geometric motifs. Often these were tilings (tessellations, mosaics), where copies of a few proto-tiles were fitted together without gaps and overlappings. In this was the artisans solved two problems together: they protected the entire surface against easy pollution and decorated it. In other cases, there were no actual tiles, but the decorations resemble tilings. From the point of view of geometry, both of them can be considered as tilings, although the technical realizations are very different. The simplest tilings are the monohedral ones where there is just one proto-tile. Equal copies of triangles, quadrangles, and centro-symmetric hexagons (including the special cases of regular triangles, squares, and regular hexagons) may obviously generate such tilings, but these are not very interesting. On the other hand, we cannot make a monohedral tiling by using a regular pentagon as a proto-tile. Moreover, a periodic (crystallographic) pattern may have just 2-, 3-, 4-, and 6-fold rotational symmetry centers (point groups), the 5-fold one is excluded. This statement has a simple geometric proof (mid 19th century), but artisans could have "discovered" it experimentally much earlier. I can imagine that these geometrical restrictions gave a special challenge for some artisans, which led to a preference for figures with non-crystallographic symmetries, including five-pointed and ten-pointed star-polygons. The artisans needed a special ingenuity to introduce these figures 'locally' into periodic patterns that have different 'global' symmetries. In some sense they broke the monotonous world of triangles, squares, and haxagons by using the artistic freedom inside the repeated element. The usage of five- and ten-pointed star-polygons also influence the lattice (or grid) that regulate the periodic repetition. Mathematically speaking, the typical lattice that features in many Islamic pattern with such star-polygons is based on the rhomb with angles of 72° (= 360°/5) and 108° (= 180° - 72°). Obviously, the construction of these complicated patterns requires a very high level of accuracy. A small "mistake" at the beginning would be "enlarged" during the repetition of units, which tendency may even interrupt the continuation. Some artisans also tried to make non-periodic patterns with a five-fold symmetry center. These topics have a link to the extreme and mean ratio (golden section): the exact construction of regular pentagons, decagons, and the described rhombs (golden rhombs) are related to this proportion. Consequently, some artisans could have adopted geometrical constructions that are based on the extreme and mean ratio. This does not mean that they definitely connected this proportion with aesthetical-philosophical ideas, but they could have used the visual appeal of some of the related figures. However, we should not over-emphasize the importance of the extreme and mean ratio in the context of design. One may easily construct a very accurate regular pentagon or a golden rhomb "visually", without using the extreme and mean ratio or any mathematical method. Even in the case of preference for geometrical constructions, artisans could have turned to approximate methods where the extreme and mean ratio is not used. In practice, a simple approximate method may produce better accuracy than a complicated exact method. Still, there is a room here for the artistic-mathematical usage of the extreme and mean ratio and it is useful to survey the related literature very briefly. Perhaps the first important question is the following one: is there any mathematical work that focuses on the regular pentagon and decagon? The answer is yes:

> Abu Kamil (ca. 850-930): *Fi al-mukhammas wa'l-mu'ashshar* (On the Pentagon and the Decagon, Manuscript in Arabic, Istanbul; Hebrew and Latin versions in Munich and Paris).

> This book discusses 20 problems and 17 of them were later adopted by Fibonacci in his *Practica Geometriae*. (Note that Fibonacci or Leonardo of Pisa, a son of a diplomat who spent a long period in Islamic countries,

introduced his famous sequence not in this book of 1220, but in the earlier *Liber abbaci*, Book of Calculations; on the other hand, it is possible that the idea of the "Fibonacci sequence" also had some sources in Islamic mathematics.) Interestingly, Abu Kamil is not keen on using the extreme and mean ratio in the case of the most simple problems. For example, he calculates the length of the side of a regular pentagon where the diameter of its circumcircle is given, but does not even mentions the extreme and mean ratio (Problem 1). On the other hand, he uses this term, referring to Euclid, all together six times in the case of three problems where he calculates the length of the side of a regular decagon with a given area and the lengths of the side and of the diagonal of a regular pentagon with given areas of triangular parts (Problems 17-19). The text of Abu Kamil's book is available in both German and English versions (cf., H. Suter's German translation of the Hebrew manuscript, *Bibliotheca Mathematica*, 3rd series, Vol. 10 (1909-1910), pp. 15-42; M. Yadgari and M. Levey's translation of the Arabic manuscript, *Japanese Studies in the History of Science*, Supplement 2, 1971). We must admit that Abu Kamil's book is rather algebraic than geometric and it is hardly useful for artisans. On the other hand, this is a relatively early work and could have inspired further studies on the pentagon and decagon.

A geometrical interest in the extreme and mean ratio is clearly demonstrated by al-Nayrizi's (or Anaritius') commentary on Euclid's *Elements* (around 900) which includes a simple 'textbook construction' of the extreme and mean ratio using a triangle with two perpendicular sides of 1 and 2; it was adopted from a work by Hero which is not extant (see M. Curtze's edition, Leipzig, 1899, pp. 107-108). Let us focus on such works where the transmission of mathematical ideas to art can be suspected. The only surviving work that emphasizes such a link, beyond some mathematical-musicological works (al-Farabi, Omar Khayyam) and mathematical-architectural calculations (al-Kashi), is the following one:

Abu 'l-Wafa' (or Abu al-Wafa') al-Buzjani (940-997/998): *Kitab fima yahtaju ilayhi al-sani' min a'mal al-handasa* (Book on What the Artisan Needs of Geometric Constructions; many manuscripts versions in Arabic in Cairo, Istanbul, Milan, Paris, and Uppsala, and in Persian in Mashhad, Paris, and Tehran; also commentaries)

This book is a masterpiece of Islamic geometry where practical problems and theoretical questions are united. Abu 'l-Wafa' presents various geometric constructions, including both exact and approximate ones, and uses the extreme and mean ratio indirectly, without naming it, several times (regular pentagons, decagons, and polyhedral structures). The lack of any clear reference to the extreme and mean ratio hints that this concept was not known in the artisans' circles. However, we cannot draw the same conclusion if we consider some later copies or commentaries of the manuscript where direct references to the extreme and mean ratio were added. There is an obvious question in connection with Abu 'l-Wafa's work and its versions: Did artisans really use this book? In the given form it is not a textbook for those artisans who had no mathematical background, but rather a scholarly work on practical geometry. However, we may interpret its title as Abu 'l-Wafa' intention of sharing his geometrical ideas with artisans. Indeed, he participated at discussions between geometers and artisans, as it is remarked in the text of the book. This statement is at least a circumstantial evidence that some artisans used the expertise of Abu 'l-Wafa', not definitely by reading his book, but by asking his personal explanations. Often mathematicians and artisans worked at the same court and they had various possibilities for discussions (cf., A. Özdural's paper "Omar Khayyam, mathematicians, and *coversazioni* with artisans", *Journal of the Society of Architectural Historians*, 54 (1995), pp. 54-71). Interestingly, the Persian version of Abu 'l-Wafa's book in the Paris manuscript is followed by another art-related survey that we shall discuss in the next paragraph.

Last, but not least, we should consider those books that indicate the geometrical interest of artisans. There are a few surviving pattern books from the 15th-17th centuries (*Topkapi Scroll*, late 15th or 16th century; *Tashkent Scrolls*, 16th and 17th centuries). Unfortunately, these works have no text with one exception:

Anonymous treatise: *Fi tadakhul al-ashkal al-mutashabiha'aw mutawafiqa* (On Interlocking Similar or Corresponding Figures, in Persian, perhaps 17th century, its original version was written sometimes in the 11th-13th centuries; the only known copy is in Paris)

If we use the modern terminology, the title can be interpreted in the following way: "On Tilings with Similar or Congruent Figures". The text explains various constructions of figures and patterns without too much theoretical details. The style of the whole work and various remarks in the text indicate that the anonymous author was, with great probability, neither a geometer, nor an artisan, but a scribe who recorded the discussions between the two groups (Özdural's interpretation). Although some experts suggest that this treatise was written as an appendix to Abu 'l-Wafa's book (Bulatov's view), perhaps the two Persian manuscripts were bounded together without such a link. At least the anonymous author does not demonstrate that he mastered Abu 'l-Wafa's work and we may even suspect that some of his sources were oral ones, if we accept that his reference to Ibn Haytham is simply the misspelling of Omar Khayyam's name (cf., Özdural's cited paper, pp. 64 and 70-71, notes 45-50). The author presents various methods for constructing regular pentagons and

> pentagrams (five-pointed star-polygons) approximately or exactly. Among these there is an unusual example: first he divides the arch of a semi-circle (half of a circular line) into five equal parts, then he constructs a regular pentagram inside the semi-circle (half of a circular disk), not in the full circle as most works do. The method is exact and based on the geometrical construction of the extreme and mean ratio (folio 186 *verso*; Bulatov's edition, Fig. 23). On the other hand, this figure has no explanation and the extreme and mean ratio is never mentioned in the book. The anonymous work is available in Russian translation as an appendix to M. S. Bulatov's book *Geometricheskaya garmonizatsiya v arkhitekture Srednei Azii IX-XV vv.* (Geometrical Harmonization of the Architecture of Central Asia in the 9th-15th Centuries, in Russian, Moskva: Nauka, 1978; 2nd ed., ibid., 1988). More recently this treatise attracted further studies (Chorbachi, Necipoglu) and a survey paper (Özdural's work appeared in *Muqarnas: An Annual of Islamic Art and Architecture*, 13 (1996), pp. 191-211).

After the historic works, let us see the problem of Islamic design from the modern side. There are many recent works that provides mathematical analyses of Islamic patterns, some of them are based purely on modern theories, while other ones are using historic records extensively. Obviously, the latter group needs more attention. It is not surprising, that one of the centers of this research was formed in Uzebekistan, earlier Uzbek Republic in the U.S.S.R. Both Rempel' and Bulatov, the leading figures of this school, use the golden section several times in their analyses of buildings and ornamental patterns. (Later we shall return to the golden sectionism in the U.S.S.R.) In Spain Antonio Prieto Vives pioneered mathematical works on Islamic patterns and demonstrated the importance of the extreme and mean ratio, although without referring to this concept (see, e.g., his paper in *Investigacion y progreso*, 7 (1933), pp. 193-206). We may say the same thing about the paper by A. J. Lee who deals with lattices based on the golden rhomb, but does not use this term ("Islamic star patterns", *Muqarnas: An Annual of Islamic Art and Architecture*, 4 (1997), pp. 182-197, see especially p. 190). The topic of the extreme and mean ratio also attracted the interest of some modern architects in Egypt and other countries where the masterpieces of Islamic art are located (see, e.g., A. A. Sultan's paper "Notes on the divine proportion in Islamic architecture", bilingual in Japanese and English, *Process: Architecture*, 15 (1980), pp. 131-163; cf., the critical comments by B. O'Kane, *Annales Islamologiques*, 26 (1992), p. 72). Summarizing our findings, it is clear that the concept of the extreme and mean ratio was occasionally used by mathematicians and some of them could have inspired artisans to use it. However, it is striking that the discussed mathematical-artistic treatises remain silent on this concept, although apply it indirectly. We may make similar statements in the case of mathematics and art in India, where even Abu 'l-Wafa's book was available and attracted mathematical commentaries.

2.3 The Latin *extrema et media ratio* (extreme and mean ratio) and the Fibonacci numbers in the European Middle Ages

The Greek term "extreme and mean ratio" and its Arabic version was translated into Latin as either *extrema et media ratio* or proportion having *medium et duo extrema* (a mean and two extremes or a middle and two ends). We may follow the usage of this concept in the translated versions of Euclid's book and a few other mathematical works, but we have difficulties to relate it to works of art. In general, we have very few clues on the geometrical ideas of medieval artists. This is also true for the "secrets" of medieval masons. Perhaps there were some "golden rules" derived from ancient traditions and these were extended by practical methods and experimental data that could have varied among different schools. The basis of their "geometry" was not the Euclidean geometry with mathematical reasoning, but an exciting "visual geometry". The artistic form of churches and other buildings could have been further refined by various "mystical facts", as it is remarked, unfortunately without details, by William of Novara, the builder of St.-Bénigne in Dijon (*Chronicon Sancti Benigni Divionensis*, early 11th century). Modern archaeologists suggested, for example, that the dimensions of the nearby Great Church at Cluny Monastery in France (its building begun in 1088) includes, among others, 13, 21, 34, 55, and 89 Roman feet (0.295 m) lengths. Here we may speak about the usage of a set of Fibonacci numbers, before Fibonacci (K. J. Conant "Mediaeval Academy excavations at Cluny, 9: Systematic dimensions in the buildings", *Speculum: Journal of Mediaeval Studies*, 38 (1963), pp. 7-11). However, I would not call it as a systematic usage of the Fibonacci numbers since their additive property does not play a role and the given numbers are related to very different structural elements (13, 55, and 89 feet are heights of different units, 21 feet is the radius of the apse, 34 feet is the width of the principal nave). Note that analyzing the vertical levels of the church at the presented longitudinal section (Plate 12 in the article), 55 appears as 15 + 40, not as 21 + 34, but here we may see a link to other Fibonacci numbers and even to the golden section: 15/40 = 3/8, the complementary part of 5/8. On the other hand, I am less convinced in the case of the appearance of the golden number at the layout, since external and internal dimensions, where the thickness of the walls are included or excluded, are compared. We may agree that the proportions of the Cluny Monastery represent an interesting case and it needs further studies.

There are many other works where the authors, on the basis of measured data, suspect that the masons used the golden section. Occasionally, they extend this statement by references that the clerical patrons were well versed in Neo-Platonic number philosophy. However, in the surviving works of Plato there is no clear reference to the golden section or its approximation, as we have seen earlier. Are there more specific arguments that may support the conscious usage of the golden section in the Middle Ages? Otto von Simson in his book *The Gothic Cathedral: Origins of Gothic Architecture and the Medieval Concept of Order* (New York, 1956) suggests, for example, that proportion is perceived as "polyphonic" at St. Michael's Cathedral of Hildesheim and as "symphonic" at the Chartres Cathedral. Specifically, in the first case the different parts of the Cathedral are treated as autonomous units, while in Chartres the ground plan and the elevations are determined by a single ratio, the golden section (p. 214). Von Simson claims that the School of Chartres knew the importance of the golden section from Euclid. Specifically, Thierry of Chartres possessed a copy of a Latin translation of Euclid's book by Adelard of Bath, moreover a Latin version of Ptolemy's *Almagest*, where a simple geometric construction of the golden section is available, was dedicated either to him or to his successor as chancellor of the school (p. 155). These are interesting arguments, but none of these two books give a special emphasis to the extreme and mean ratio (golden section). Many scholars had access to the same books through the ages, but did not initiate the usage of the extreme and mean ratio in architecture. Interestingly, even some art historians who refer to the golden section in other cases do not fully accept von Simson's view. For example, N. Luning Prak remarks that although a Neo-Platonic influence of the cathedral school on the plan of Chartres is probable, von Simson fails to demonstrate his thesis with the plans and elevations ("Measurements of Amiens Cathedral", *Journal of the Society of Architectural Historians*, 25 (1966), p. 212).

A totally different approach is initiated by Peter Kidson, a historian of art in Britain. He suggests considering the proportional problems in the context of metrology because this field may demonstrate the mathematical principles more clearly than the measured data of concrete buildings ("Metrological investigations", *Journal of the Warburg and Courtauld Institutes*, 53 (1990), pp. 71-97). First of all, Kidson explains that there were simple methods for the rational approximation of irrational numbers, including the golden number, from ancient times. Then he demonstrates that the ratio of two types of foot, which were used in the German cities Rostock and Schwerin respectively, is 5/8. Of course one may say that this is just an arithmetical ratio. However, the German name of the Schwerin foot *geometrische Kettenfuss* (geometrical chain foot) indicates a link beyond simple arithmetics. We may agree that the golden section should be a strong candidate to associate the ratio 5/8 with geometrical ideas. On the other hand, one may have some objection. Indeed, we cannot exclude the possibility that geometry simple refers to "earth measurement", which is the original meaning of this word. In addition to this, Kidson demonstrates his thesis with 18th century documents by adding that perhaps the medieval name and ratio were the same. The author also considers the ratio of the Roman foot to the average of the 14 different versions of the *piede Liutprando*, which is named after Liutprand, the King of the Lombards. Again this ratio is close to 5/8. Obviously, this is a remarkable hypothesis that needs further study, including the consideration of archive materials on medieval metrology. In short, we may conclude that the extreme and mean ratio could have had some importance in the Middle Ages, but we are not aware of a convincing evidence.

2.4 The Italian *divina proportio* in the Renaissance: a new view on the Leonardo-Pacioli connection

The extreme and mean ratio gained a special attention from two scholars who popularized Euclid's *Elements* in Latin:

- Campanus (13th c.), who prepared a Latin manuscript of Euclid's book on the basis of earlier translations, claimed that the extreme and mean ratio is worthy of philosophers' admiration,

- Pacioli (15th-16th cc.), who revised and extended Campanus' work, renamed the extreme and mean ratio as the "divine proportion" in another work and published an entire book in Italian entitled *Divina proportione* (Venice, 1509).

The fact that the latter deals not only with mathematical, but also artistic questions, as a long subtitle emphasizes, and it includes, although in a subtle form, contributions by both Leonardo da Vinci and Piero della Francesca, two leading artists of the Renaissance, led to the widespread 'legend' that Pacioli suggested using the extreme and mean ratio (divine proportion) in art. The confusions start at the very title of the book, which is derived from its first part and does not represent the whole work in the given mathematical sense. Specifically, Pacioli's book is the compilation of four earlier manuscripts:

(1) "Compendio de divina proportione", where he discusses the mathematical properties of the extreme and mean ratio according to Euclid's theorems with various mystical additions and extends the study on the five regular polyhedra (Platonic solids) by semiregular and stellated ones;

(2) "Tractato de l'architectura", where he does not suggest using the divine proportion in the earlier mathematical sense, but following Vitruvius here rather than Euclid, he prefers simple ratios;

(3) the Italian translation of Piero della Francesca's Latin work on the five regular polyhedra, without giving a proper credit to him;

(4) an appendix on the proportions of Roman letters (which topic is related to Part 2).

Note that Piero's geometrical manuscripts also include some interesting calculations related to the extreme and mean ratio, but it is not clear how this approach could have infected his paintings.

Leonardo da Vinci, who drew the polyhedra for Pacioli's manuscript *Compendio de divina proportione*, could have had some interest in the divine proportion (extreme and mean ratio), but, contrary to the frequent statements in modern works, only a limited one. In his *Treatise on Painting* (*Codex Urbinas*), which was compiled shortly after his death, he uses the expression "divine proportion" three times, but without explaining its meaning. All of the three references are in an introductory part to the whole treatise, the *Paragone* (comparison or rivalry of the arts). The tradition of similar expressions is going back to ancient times. Thus, Plato referred to "divine harmony" (*theia harmonia*) in music, which gives pleasure to everyone and also delight to intelligent people (*Timaeus*, 80B). Since Greek scholars discussed the musical harmonies in terms of geometrical ratios of a vibrating string, we may say that Plato almost outlined the idea of "divine proportion" in a general sense. In the Renaissance the strong influence of theology and Neo-Platonic thought made the adjective "divine" very popular: Dante spoke about "divine nature", Bocaccio referred to "divine intellect", Petrarca meditated on "divine beauty", and Dante's *La Commedia* (14th c.) was renamed *La Divina commedia* (16th c.). Leonardo also frequently employed this adjective: his "divine proportion" is associated with "divine beauty" (Chaps. 28 and 42) and "divine harmony" (Chap. 41, McMahon edition). According to Leonardo, the true understanding of the forms of nature is related to the understanding of the Creator (cf., Chap. 80). This view explains his preference for the adjective "divine". Although it is widely believed that Pacioli originated the expression "divine proportion", I have a different conjecture (which will be discussed in more detail in another paper):

- 1485-1490: First Leonardo used the expression "divine proportion" in a simple metaphorical sense as good proportion (similar to Plato's "divine harmony"). My dating is based on one of three usages of the expression, which is associated with a historic figure, Matthias Corvinus of Hungary (b. 1443, d. 1490): "Reply of King Mathias (sic) to a poet who competed with a painter", Chap. 28 (here I challenge Carlo Pedretti's suggested date of ca. 1492).

- 1494: Pacioli, a Franciscan monk and mathematician, used the expression *proportione auete el mecco e doi extremi* (proportion having a mean and two extremes) in his book *Summa de arithmetica, geometria, proportioni et proportionalita* (printed in Venice). This work became very popular and significantly contributed to the popularization of the double-entry book-keeping.

- 1496-1497: Pacioli adopted Leonardo's expression "divine proportion" soon after they met and their close cooperation began (the alternative possibility is that Pacioli had the same expression in mind independently, but even in that case Leonardo could have inspired him to use it).

- 1497: Pacioli as a mathematician gave a new specific meaning to this expression and eventually transformed it into a scientific term that gives an alternative to the complicated ancient phrase "proportion having a mean and two extremes" (*Compendio de divina proportione*, with Leonardo's illustrations of the polyhedra, 1497). Pacioli's manuscript also includes the earliest known reference to a version of Leonardo's *Treatise*, consequently we may suspect that Pacioli was aware of Leonardo's expression "divine proportion".

- 1509: Pacioli got a special privilege to print books and he united four manuscripts under the general title *Divina proportione* (Venice, 1509). Leonardo's drawings were replaced by relatively low quality woodcuts. In the book there is no direct credit to Leonardo for the original drawings, but Pacioli refers to his contributions elsewhere in the text.

We may understand the title of Pacioli's printed book much better in the above context. Indeed, the expression *divina proportione* is not so misleading in the title if we consider both the strict mathematical meaning (Parts 1 and 3 of the book) and the broader metaphorical interpretation (Parts 2, 4, and some mystical statements in Part 1). On the other hand, Pacioli did not prefer to use this term in Parts 2, 3, and 4 (with the exception of cross-references to Part 1), which fact could have had historic reasons. Indeed, in Part 2 he frequently refers to the Vitruvian term *symmetria* and in Part 3 he uses, following Francesca (and Euclid), the expression "proportion having a mean and two extremes". The latter is also employed in Pacioli's edition of Euclid, basically a revision of the earlier work by Campanus. It was printed in 1509, the same year that the book *Divina proportio* was published. Although Pacioli added various comments to the Euclidean text, he did not refer to his new term "divine proportion". Why was he so shy in this case? What is the reason for this strange fact? I can imagine that there are psychological motives here, an unusual way of giving credit to Leonardo. Pacioli could have had a feeling that the topic of the "divine proportion", including both the very term and the study of the polyhedra, was a joint work with Leonardo. Thus, he did not prefer this term beyond the manuscript of 1497, when they actually cooperated, and its later printed version.

Our proposed reconstruction of the development of the term "divine proportion" also gives a possible explanation for another question: Why did Leonardo abandon his initial use of the expression "divine proportion" in other parts of his *Treatise*, beyond the cited three chapters in the introductory part *Paragone*? Indeed, he continued to write his *Treatise* until around 1513, but even in the course of the detailed discussion about proportions he preferred other phrases ("well proportioned man", "natural proportion", Chaps. 285-286). I think that Leonardo was aware of the new mathematical meaning of his earlier expression and perhaps did not want to limit himself to that special case. Although some of Leonardo's statements, similar to the case of Vitruvius, may have an association with the main idea of the extreme and mean ratio (e.g., "every part of the body should in itself be in proportion to the whole body", Chap. 287), the strict mathematical concept is not introduced in this *Treatise*. Note that Leonardo was also interested in another topic that can be linked to the extreme and mean ratio: the arrangement of leaves on a stem (Chaps. 884-889). We may say, however, that his observations represent an interesting step to the discovery of a geometric tendency in this field, but it is far from the strict description of the botanical phenomenon that is associated with Fibonacci numbers (phyllotaxis).

2.5 The missing *sectio aurea* (golden section) in Leonardo's manuscripts (?)

Many modern artists and scholars, including the founders of the cubist group *Section d'or*, claim that Leonardo used or even coined the term *sectio aurea* (golden section). However, we could not locate this expression in Leonardo's surviving treatises and other Renaissance works. Of course this is not a proof that the term was not used. Interestingly, most of the claims that Leonardo coined the term *sectio aurea* do not give any concrete reference to Leonardo's manuscripts, while others point to his works on human proportions, especially to the famous proportional drawing of a male body that is inscribed into both a circle and a square, *homo ad circulum* and *homo ad quadratum*, and referred to as 'Vitruvian canon' or simply 'Vitruvian man' (its original is at the Galleria dell'Accademia, Venice). The center of the circle is the navel. However, contrary to many other attempts to illustrate a partly unclear statement by Vitruvius (*De architectura*, Book 3, Chap. 1, Para. 3), the square is not concentric with the circle, but shifted to a lower position. On the other hand, Leonardo does not say that the navel divides the height according to the golden section. His text accompanying this drawing, similar to the statements by Vitruvius, refers to ratios of small integers, as well as to an equilateral triangle, but not to the Italian *sezione aurea* or related terms. Very recently we have found a new opportunity to check the Italian spread of this expression in one of the largest historical-etymological dictionaries ever prepared, in the *Grande dizionario della lingua italiana* (by S. Battaglia, Vols. 1-18, Torino, 1961-1996, still incomplete) that finally reached the entry "Sezione aurea" (Vol. 18, *Scho-Sik*, 1996, p. 830; cf., the shorter entry at "Aurea", Vol. 1, *A-Balb*, 1961, p. 847). This dictionary supports our view that Leonardo did not introduce this expression, since there is no reference to him; moreover, the Italian term is derived from a 19th century German expression and a later French one. (In the case of *sezione* itself, the dictionary gives credit to Renaissance artists, including Ghiberti.)

Of course the lack of this term does not mean that Leonardo did not use the concept of the golden section in the case of the position of the navel and some other details in his drawings. However, Leonardo's surviving proportional calculations and sketches with regulating lines indicate a strong preference for simple ratios of integers. The art historian R. Wittkower pointed out impressively that Renaissance artist had a great interest in the empirical procedure of measuring and consequently in commensurable proportions (*Architectural Principles in the Age of Humanism*, New York, 1971, Appendix 2). Although this relationship seems self-evident at first glance, we may have some objections. First of all, any practical measurement is approximate and thus we may attempt at both commensurable and incommensurable ratios alike. Thus, it is not easier to measure, let us say,

$$1/7\,d = 0.142857...\,d \quad \text{than} \quad \sqrt{2}\,d = 1.414213...\,d$$

where d is a given distance.

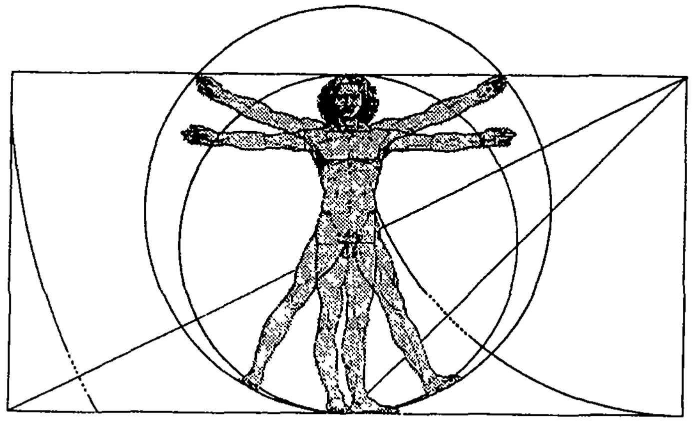
We admit that this argument is beyond Renaissance thinking, since the calculation of root numbers were not widely known. Note, however, that Alberti, who laid down the theoretical basis of the Renaissance in his Latin and Italian works, remarked that "there are some other natural proportions for the use of structures, which are not borrowed from numbers, but from the roots and powers of squares" (Ten Books on Architecture, p. 199; cf., P. H. Scholfield *The Theory of Proportion in Architecture*, Cambridge: Cambridge University Press, 1958, p. 51). Alberti was a humanist with a very broad knowledge, thus, one may say, we should not overemphasize this quote. I have, however, another point against Wittkower's thesis: the combination of measurements and simple geometric constructions may provide *an easier and better approximation of some incommensurable line-segments than of commensurable ones*. For example, one may easier construct $\sqrt{2}\,d$ (a diagonal of a square with the side d) or $\sqrt{3}\,d$ (the height of an equilateral triangle with the side of $2d$) than $1/7\,d$ (either 'measuring' $1/7\,d = 0.142857...\,d$ or constructing this ratio with an arbitrary unit and then projecting it parallel to the given line-segment). In spite of this, I agree with Wittkower that the surviving historic records show a preference for commensurable ratios. Still, we should study some special cases, including Leonardo's sketches that are associated with geometrical constructions. Indeed, in those cases we may have more hope for detecting the usage of the golden section in the 'secret geometry' of pictures than in the case of actual systems of human proportion.

There is a topic that we should discuss here. Vitruvius referred to the diagonal of a square (Book 4, Chap. 1, Para. 11; Book 6, Chap. 3, Para 3). Filarete used the diagonal of a double-square (*a due quadri il diamitro*) and Francesco di Giorgio Martini introduced three module systems and one of them is based on the same diagonal (cf., H. Saalman "Early Renaissance architectural theory and practice in Antonio Filarete's *Trattato di architettura*", *Art Bulletin*, 41 (1959), pp. 91-92; H. Millon "The architectural theory of Francesco di Giorgio", *Art Bulletin*, 40 (1958), p. 258). At this point we are very close to the construction of the golden section:

- the diagonal of a double-square (two equal squares in juxtaposition, i.e., placed side by side) is $\sqrt{5}$, where the side of a square is 1 unit-length;

- the golden section is $(\sqrt{5} - 1)/2$, i.e., we should subtract 1 unit-length from the diagonal and then halve the remaining line-segment.

Both Filarate and Francesco were senior to Leonardo. We also know that Leonardo owned some of the manuscript of Francesco. Leonardo also dealt with a system of proportion based on a double-square (which is briefly mentioned on folio 12,304 *recto* in the Royal Collection at Windsor).



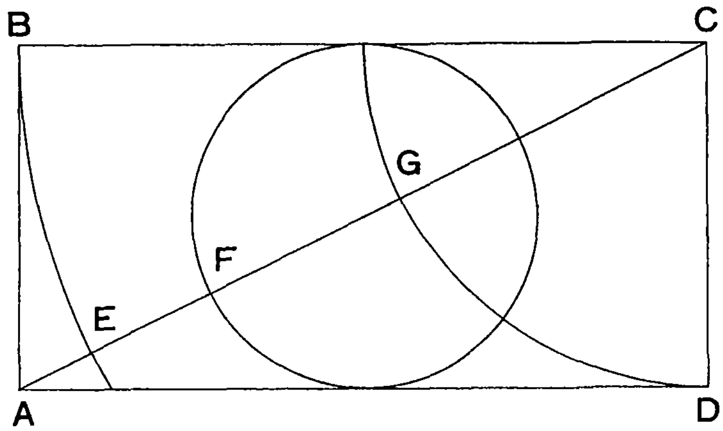


Figure 2: Mukaigawa connects the cited 'Vitruvian canon' with a seemingly independent geometrical figure in Leonardo's *Codex Madrid* (Vol. 2, folio 81 *recto*) via another geometrical sketch in the *Codex Huygens* (folio 1). The latter is an anonymous manuscript which was influenced by Leonardo (cf., E. Panofsky *The Codex Huygens and Leonardo da Vinci's Art Theory*, London, 1940; Reprint edition, 1976; also see S. Marinelli's survey about the possible authors, *Journal of the Warburg and Courtauld Institutes*, 44 (1981), pp. 214-220). Since the sketch in the Codex Huygens is based (approximatively) on a double-square where a diagonal is intersected by appropriate arches, there is a series of golden section points on this line: see G on AC, F on AG, and E on AF. Our data on Filarete, Francesco di Giorgio Martini, and even Leonardo's interest in the double-square give some support to his theory. On the other hand, there are some chronological problems that we shall discuss. S. Mukaigawa's detailed study is available in his paper "Reonarudo no 'jintai kenkō zu' kenkyu: Sono 'en' to 'seihōkei' ni tsuite, [A study of Leonardo's 'scheme of human proportion': About its 'circle' and 'square', in Japanese with an English summary], *Bijutsu-shi*, No. 129 [Vol. 40, No. 2] (1991), pp. 98-113.

Is there any evidence that Leonardo constructed the golden section in his sketches? The only attempt known to us where the possible usage of the golden section is supported by some geometrical evidence is a Japanese work by S. Mukaigawa. Specifically, he suggests that in the case of the 'Vitruvian canon' the golden section could have played a role during the process of constructing the relative position of the circle and the square around the male body (Fig. 2). It is true that one may have an obvious objection against this theory on a chronological basis: Leonardo's proportional drawing was composed around 1490 (this dating is based, however, on the style of drawing and the type of handwriting, not on a strict evidence!), long before his first meeting with Pacioli in 1496 and the cited geometrical studies of the *Codex Madrid* in 1503-1504. Still, we should not automatically reject Mukaigawa's theory, even if we do not challenge the dating of the proportional drawing. By 1490 Leonardo could have had some knowledge about the extreme and mean ratio and related geometrical constructions from Euclid's book, specifically from Campanus's Latin translation with the praise of the extreme and mean ratio, which was first printed in Venice in 1482. Indeed, Leonardo's book-list in the *Codex Madrid* (1503-1504) makes clear that he owned "*euclide in geometria*", but we do not know which edition.[2]

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[2] C. Pedretti gave a detailed analysis of this book-list and in the case of this item he remarked that the editions of Euclid's *Elementa geometriae* that Leonardo could have owned are: Venice, 1482, Vicenza, 1491, and Venice, 1502 (cf., his *Commentary* to Richter's anthology *The Literary Works of Leonardo da Vinci*, Vol. 2, Berkeley, 1977, p. 357). However, this list is incomplete and misleading. The correct list should be: *Praeclarissimum opus elementorum Euclidis [...] in artem geometriam [...]*, Venice 1482 (two versions); Ulm, 1486; Vicenza, 1491; and G. Valla's *De expetendis et fugiendis rebus*, Venice, 1501, which is an encyclopedic work that includes, among others, a number of theorems and proofs from Euclid (see, e.g., Sir Thomas Heath *The Thirteen Books of Euclid' Elements*, Vol. 1, New York, 1956, pp. 97-98). The item of 1501 can be excluded since it is not "Euclid's geometry".

We also suggest extending Mukaigawa's theory with the study of two further topics:

(1) Leonardo's system of proportion based on a double-square, which is mentioned on folio 12,304 *recto* in the Royal Collection at Windsor Castle (cf., Richter's cited anthology, item no. 337, and Pedretti's related commentary).

(2) The Oxford sheet entitled *Simetria del corpo humano* (Proportions of the Human Body, in Italian), Christ Church Library, Oxford, inventory no. 0012, which is probably related to the *Codex Huygens*. It presents an additional approach to the problem of the relative position of the circle and the square around the male body (cf., Pedretti's cited *Commentary*, Vol. 1, pp. 249-251).

Note that M. Robertson, giving credit to H. W. Roberts, presents a different theory about the same geometrical problem. His approach is associated with the diagonals of some golden rectangles and a theorem of Euclid ("The golden section or golden cut", *Journal of the Royal Institute of British Architects*, 55 (1948), pp. 536-538). Incidentally, the Russian architect I. P. Shmelev has a surprising view on the "Vitruvian man": he claims that the closed and the out-spread legs have different lengths, consequently Leonardo's drawing is not serious, but a "joke" (see Shmelev's book, with co-authors, *Zolotoe sechenie: Tri vzglyada na prirody garmonii*, The Golden Section: Three Views on the Nature of Harmony, in Russian, Moskva, 1990, p. 291). Of course the last view is controversial: the quality of the drawing and the accompanying text indicate a serious attempt and the mentioned differences in length may have simple anatomical reasons.

Another possibility is that Leonardo did not use the golden section in 1490, but later returned to the geometrization of this problem and introduced it. Although we admit that Leonardo's surviving manuscripts indicate neither a systematic study of geometrical problems prior to 1490 (with the exception of some sheets in *Manuscript B*, around 1489, and *Manuscript C*, around 1490), nor a later reconsideration of the discussed proportional drawing, we should not forget that about two-thirds of Leonardo's writings have been lost. Of course we cannot exclude the possibility that there were no further data because Leonardo had no desire to reveal his secrets.

2.6 Kepler's *sectio divina*: the German case from Dürer's geometrical works to Kepler's astronomical discoveries

It is also interesting to take a look at the case of Albrecht Dürer, the 'German Leonardo' (being half-Hungarian from his father's side), who was not only a leading artist of his age, but also published theoretical books on mathematics, fortification, and human proportions, respectively, in German (Nuremberg, 1525, 1527, and 1528). Luckily the larger part of his manuscripts survive. Dürer had a direct link to the Italian Renaissance: he visited Venice in 1494-95 and 1505-1507. At the very end of this second period, he made an extra trip to Bologna to learn the "secret of perspective", as he claims in a letter. Many researchers suggest that Pacioli was the unnamed person whom he planned to visit and this story is tied to Dürer's new interest in the golden section. Let us look at both of these statements. Although we cannot exclude the possibility that Pacioli was his 'teacher', there are many other possibilities, including practicing artists. I also believe that the meeting of two relatively famous persons would have led to some records by the participants or their acquaintances. However, no such records survive. It is especially striking that the books and the surviving manuscripts by Dürer do not refer to Pacioli and do not use the term "divine proportion" or any alternative expression for the golden section. Incidentally, the possible meeting of 1507 would have taken place two years before the publication of Pacioli's book. Of course, despite this fact, Dürer could have discussed the topic of the divine proportion with Pacioli. On the other hand, Dürer's works and manuscripts after this trip do not reflect such an interest. Those documents that some researchers associate with the golden section were written around 1513. At that time Dürer made several notes on the proportions of the human body. Considering the three longest parts of the human body (*trey lengsten teill des mans*), specifically

- the torso (from neck to waist),
- the two thighs (from hip to knee),
- the two shins (from knee to ankle)

he compares the two shorter ones to the longest (*zwu myndern leng gegen der gröseren ersten vergleichen*) and concludes that the first is proportional to the second as the second to the third (London Manuscripts, No. 5230, folio 61 *recto*; see this and three similar statements in *Dürer: Schriftlicher Nachlass*, ed. by H. Rupprich, Vol. 2, Berlin, 1966, pp. 184-190; an English translation of the quote is available in W. L. Strauss's book *The Complete Drawings of Albrecht Dürer*, Vol. 5, New York, 1974, p. 2475). In this quote we have nothing more than a geometric mean

$$a/b = b/c$$

where a, b, c are the lengths of a shin, a thigh, and the torso, respectively. It is not required that the sum of the two shorter parts is equal to the longest ($a + b = c$). On the other hand, Dürer adds a brief note: Euclid states (*Ewclides setzt*). Unfortunately, there is no further detail. This fact made room for the conjecture that this is a reference to the extreme and mean ratio (golden section) in Euclid's book. However, this is not conclusive evidence: Euclid's book also deals with the geometric mean in detail. Moreover, I propose a simple biological argument against the possible use of the golden section: the necessary equation is impossible since

$$a + b > c \quad \text{(i.e., shin + thigh > torso).}$$

Dürer's proportional drawings of the same period deal with simple ratios of small integers and there is no preference for ratios of neighboring Fibonacci numbers. Note that Dürer had no intention to create just one ideal canon, but he considered several varieties. His systems of proportions (plural!) represent a step from artistic work to scientific anthropometry. Unlike Leonardo's scientific works that remained hidden in unpublished manuscripts for a long period, Dürer published his main ideas in printed books, including a mathematical monograph on regular polygons and polyhedra, conic sections, and other topics. It became an influential work and later, among others, Kepler also studied it and referred to the approximate construction of a regular heptagon described by Dürer (cf., *Harmonice mundi*, in Latin, Linz, 1619, p. 39 = Kepler's *Gesammelte Werke*, Vol. 6, Mümchen, 1940, p. 55). Considering the body of Dürer's published works, we may claim that he, similar to Leonardo, had no special interest in the golden section, although we cannot exclude a minor involvement.

There is one more related topic that we should consider here: the Renaissance interest in regular pentagons. Indeed, the *Codex Huygens*, which is associated with Leonardo's circles, includes a proportional drawing (folio 7) where the human body is inscribed not only into a square and a circle, but also into a pentagon that only slightly differs from a regular one. The head and the four limbs, hands and feet outspread, may easily inspire a pentagon or even a pentagram (five-pointed star). The philosopher Agrippa von Nettesheim also has such an illustration in his book *De occulta philosophia*. In the late 15th century and early 16th century some artists and architects (master builders, masons) described various approximate methods for the geometrical construction of a regular pentagon, including M. Roriczer, Leonardo, and Dürer. On the other hand, these approximate constructions avoid using the golden section. In addition to this, most pentagon-related artistic studies did not give a special role to the points of intersection of the diagonals and thus did not indicate any clear commitment to the extreme and mean ratio. We may have a better chance identifying the use of this proportion in those cases where, beyond considering an exact (not approximate!) regular pentagon, some related calculations or geometrical proofs are involved.

The case of the German astronomer Kepler (1571-1630) requires special attention. He used the extreme and mean ratio during his geometrical-astronomical calculations and referred to it as

- *extrema et media ratio*, the Latin version of Euclid's expression, cf., *lineam extrema et media ratione sectam*,

- *proportio divina*, the Latin version of Pacioli's *divina proportio* which was popularized by Petrus Ramus,

- *sectio divina*, a newer term uniting *sectio* from Euclid's longer expression and *divina* from Pacioli and Ramus.

Indeed, Kepler could not have missed the concept of the extreme and mean ratio during his very first model of planetary orbits, which was based on concentric circles obtained by intersecting a set of spheres inscribed into or circumscribed around the five regular polyhedra ('nest of spheres and polyhedra', 1596). Although this model was incorrect, it opened the way for the discovery of Kepler's laws where he replaced the circles by ellipses. Thus, Kepler's enthusiasm is understandable when he compared the Pythagorean theorem and the extreme and mean ratio with a gold-nugget and a gemstone (*Mysterium cosmographicum*, 2nd edition, 1621, Chap. 12). Although, strictly speaking, the Pythagorean theorem is the gold and the extreme and mean ratio is the gemstone, we suspect that this statement contributed to the formation of the later term "golden section". On the other hand, Kepler never devoted a long chapter to the extreme and mean ratio in his books, and the larger part of his related notes are in letters.

The paper will be continued in a later issue.