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THE MATHEMATICS OF SYMMETRY: A TUTORIAL

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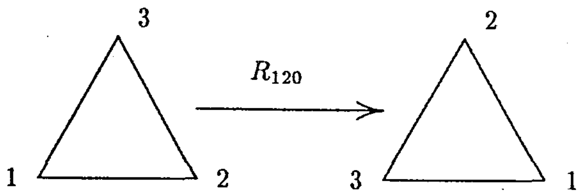
INTRODUCTION

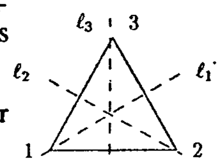
In mathematics (and physics) the concept of symmetry is expressed in terms of the action of a group, called the symmetry group, on a set. This article presents a brief tutorial on these concepts that assumes no previous knowledge of mathematics, and considers some possible extensions.

SYMMETRIES AND GROUPS

For simplicity we will consider a figure in the plane that has several symmetries: an equilateral triangle. The vertices will be labeled to allow us to keep track of the symmetry operations. These operations will be motions of the plane that transform the triangle into the triangle (i.e., leave the triangle invariant). The first motion is rotation by 120 degrees counter-clockwise, which we will denote by R_{120}. The triangle is transformed into itself, with the vertices shifted:



There are three rotations that leave the triangle invariant: through 120 degrees, 240 degrees, and 360 degrees (or, equivalently, through 0 degrees). These rotations can be combined, since a rotation through θ degrees followed by a rotation through ϕ degrees is equivalent to a rotation through $\theta + \phi$ degrees. Symbolically we will write $R_{\phi+\theta}\} = R_\phi * R_\theta$, where R_θ is applied first, followed by R_ϕ.



Reflections across lines bisecting the vertices give other symmetries. The lines will be labeled l_1, l_2, and l_3:

The three symmetries of reflection across these lines will be denoted by L_1, L_2 and L_3. Each of these motions is its own inverse, e.g., $L_1 * L_1 = R_0$.

This gives 6 motions of the plane that preserve the equilateral triangle. Combining any two of these operations gives another one of these operations. This is shown in the following table, where the operation with label in the left-hand column is performed after the operation with label in the top row.

$*$	R_0	R_{120}	R_{240}	L_1	L_2	L_3
R_0	R_0	R_{120}	R_{240}	L_1	L_2	L_3
R_{120}	R_{120}	R_{240}	R_0	L_3	L_1	L_2
R_{240}	R_{240}	R_0	R_{120}	L_2	L_3	L_1
L_1	L_1	L_2	L_3	R_0	R_{120}	R_{240}
L_2	L_2	L_3	L_1	R_{240}	R_0	R_{120}
L_3	L_3	L_1	L_2	R_{120}	R_{240}	R_0

Thus L_1 followed by L_2 gives the same result as R_{240}, so $L_2 * L_1 = R_{240}$. A set with these properties, including inverses and identities, is called a *group*. The formal definition is:

Groups

A *group* consists of a pair $< G, * >$, where G is a set and $*$ is an operation on pairs of elements of G satisfying:

(1) If a and b are any two elements in G, then $a * b$ is in G (the Closure property).

(2) If a, b, and c are any three elements in G, then $a*(b*c) = (a*b)*c$ (the Associative property).

(3) There is a special element e (the Identity element) in G such that if a is any element in G then $e * a = a * e = a$ (an Identity exists).

(4) If a is any element of G then there exists an element a^{-1} (the Inverse of a) such that $a * a^{-1} = a^{-1} * a = e$ (Inverses exist).

A group with exactly n elements in G is said to be of *order n*. A group is said to be *infinite* if the set G is infinite. A group is said to be *abelian* if the commutative property holds: for any two elements a and b in G, $a * b = b * a$.

Note that the group of symmetries of the equilateral triangle is a finite group of order 6. Since $L_1 * L_2 = R_{120} \neq R_{240} = L_2 * L_1$, the group is not abelian. This group of symmetries of the equilateral triangle is called the *dihedral* group, and is denoted by D_3.

Generalizing the table of operations of the group of symmetries of the equilateral triangle, the *Cayley table* of any group $< G, * >$ is the table of operations $a * b$:

$*$	$\cdots$	b	$\cdots$
$\vdots$			
a	$\cdots$	$a * b$	$\cdots$

Example

Let G be the set of integers and $\ast$ be the operation of addition, $+$. Then $< G, + >$ is a group. The identity element is 0 and the inverse of an integer n is the integer $-n$. $< G, + >$ is an infinite abelian group. (The Cayley table is too large to fit in the space allowed by the editors.)

Example

Let G be the set of integers and $\ast$ be the operation of multiplication, $\ast$. Then $< G, \ast >$ is not a group. It has an identity element, 1, but 2 has no inverse in G, since 1/2 is not an integer. The same is true if G is the set of non-zero integers or of positive integers: inverses do not exist. Exercise: show that if G is the set of positive rational numbers, then $< G, \ast >$ is an infinite abelian group.

Notice that in the Cayley table of the symmetries of the equilateral triangle the first three rows and first three columns of rotation operations form a sub-table that satisfies all the group properties:

$\ast$	R_0	R_{120}	R_{240}
R_0	R_0	R_{120}	R_{240}
R_{120}	R_{120}	R_{240}	R_0
R_{240}	R_{240}	R_0	R_{120}

The symmetry of the Cayley table about the main diagonal from upper left to lower right corresponds to the fact that this group is abelian: $R_\theta \ast R_\phi = R_\phi \ast R_\theta$ for any angles θ and ϕ.

A subset of a group that is itself a group (under the same operation) is called a *subgroup* of the original group.

GROUPS ACTING ON SETS

A group $< G, \ast >$ acts on a set S if for any element a in G and any element t in S there is an corresponding element $a(t)$ in S such that the following hold:

(1) For any two elements a and b in G, and any element t in S, $(a \ast b)(t) = a(b(t))$.

(2) If e is the identity element of G, then for any element t in S, $e(t) = t$.

The symmetries of the equilateral triangle, and its subgroup of rotations, act on the plane, as well as on the triangle.

Example

Let $S = \{1, 2, 3\}$ and let G be the permutations (orderings) of S:

$G = \{(1, 2, 3), (1, 3, 2), (2, 1, 3), (2, 3, 1), (3, 2, 1), (3, 1, 2)\}$

We can consider the elements of G as functions that act on S by defining

(m, n, k) applied to 1 to be m: $(m, n, k)(1) = m$;
(m, n, k) applied to 2 to be n: $(m, n, k)(2) = n$;
(m, n, k) applied to 3 to be k: $(m, n, k)(3) = k$.

The group operation $*$ is then composition of functions, so that, for example,

$$((1,3,2) * (2,3,1))(1) = (1,3,2)((2,3,1)(1)) = (1,3,2)(2) = 3$$

Under this operation G is a group: it is closed, (for example, $(1,3,2) * (3,2,1) = (2,3,1)$); associative; the identity element is $(1,2,3)$; and each element has an inverse (for example, the inverse of $(1,3,2)$ is $(1,3,2)$).

If the set $S = \{1,2,3\}$ is used to label the vertices of an equilateral triangle, then the operation $(3,1,2)$ can be viewed as rotation counterclockwise through 120 degrees. Similarly, $(2,1,3)$ reflects across a vertical line. Thus the group G is essentially the same as the dihedral group D_3 of symmetries of the equilateral triangle. G and D_3 are said to be *isomorphic*, since their Cayley tables are identical after a change of names of the group elements.

PROSPECTS

The preceding material on groups and groups acting on sets can be found in any book on group theory, (such as Durbin, 1985). The mathematical concept of group can be generalized in several ways; the one most likely to be applicable to symmetry is that of *groupoid*. A groupoid can be thought of as a group with several identities. A formal definition can be given similar to the above definition of group, but it is more complex (see Brown, 1989). A more concise definition can be given using the concepts of category theory (see Higgins, 1971): a groupoid is a category in which every morphism is an isomorphism.

Groups are commonly studied in undergraduate mathematics programs, but groupoids have remained a specialist topic, used in such fields as Lie groupoids and C^*-algebras. For groupoids to gain a place in the undergraduate curriculum, applications of groupoids to problems of symmetry, analogous to the applications of groups to symmetry, are probably required. Such applications could both enrich the mathematical theory of groupoids and provide solutions to problems in the analysis of symmetry.

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