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ON GEOMETRIC SYMMETRIES AND TOPOLOGIES OF FORMS

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ABSTRACT: *A brief bird's eye overview is given about possible symmetries and topologic structures of bodies in the 3 dimensional Euclidean space. This is done for basis of understanding some further papers of this issue.*

1. INTRODUCTION

The notion of *symmetry* is widely used for characterizing patterns of biological objects. However, it is often used purely qualitatively. It is not so in mineralogy and crystallography, but their symmetries are almost exclusively *discrete*. Crystallographic analogons are very useful in many cases (e.g., for the radial symmetry of *Ctenophora*, radial pseudosymmetry of *Echinodermata*, left-right symmetry of higher evolved animals, etc.). Still, *continuous* symmetries are also possible and they need another approach.

There are more complicated cases as well. Is an *Ammonita* symmetric? Its chambers grow along a spiral line in a regular manner. There is no left-right symmetry, neither radial symmetry, nor rotational symmetry. Still a regularity is clearly seen. Is it a symmetry, and if it is, in what sense?

The question cannot be fully answered. Any well-defined transformation may be called a symmetry. However, one knows what is a *geometrical* symmetry. The present paper applies the formalism to situations relevant for biological objects.

Section 2 is a brief recapitulation of the Killing vector technique of Riemann spaces. Section 3 applies the equations in the 3 dimensional Euclidean space. First all the Killing vectors (including conformal ones) of this space are calculated and then, now excluding the conformal ones, all possible cases are listed when the actual symmetry of the matter distribution is smaller. Section 4 contains some remarks for discrete symmetries.

Section 5 illustrates the possible subgroups of continuous symmetries on examples, while Section 6 mentions some principles of topologic classifications of objects.

Some explanations and commenting remarks, which might also properly be footnotes as well, are included into the text, but for showing the specific rôle in CAPITALS.

2. ON KILLING SYMMETRIES

This Chapter is a brief recapitulation of the theory of continuous symmetry transformations for pedestrians. For more details see (Eisenhardt, 1933) and citations therein.

Consider a Riemann (pseudo-Riemann) space of n dimensions. There is a coordinate system $\{x^i\}$ on the manifold. Because of the Riemannian structure *any* regular recoordination is permitted (Eisenhardt, 1950), but this possibility will be mainly ignored in this paper.

A domain of the space is filled with matter. This matter is characterized by a set of fields $\{\Phi_\alpha(x^i)\}$. This set contains the relevant data of the matter, say Φ_1 may be the density, Φ_2 the colour, Φ_3 classifies the tissue at x^i as liver, muscle, kidney, etc. The domain is generally finite, but we restrict ourselves to the interior and neglect the boundaries. For the present we ignore the discontinuous nature of the matter in biological objects. Then the cellular structure is averaged away. This is satisfactory if the number of cells is $>> 1$.

Now identify a number of points in the matter; 3 will be sufficient, in an infinitesimal neighbourhood

$$x_I{}^i = x^i + \mathrm{d}x_I{}^i$$

In a Riemann space the distances of infinitesimally close points can be written as

$$\mathrm{d}s_{IK}{}^2 = g_{rs}(x^i)\mathrm{d}x_{IK}{}^r\mathrm{d}x_{IK}{}^s; \qquad \mathrm{d}x_{IK} = \mathrm{d}x_I - \mathrm{d}x_K \tag{2.1}$$

(NOTE THE EINSTEIN CONVENTION (Eisenhardt, 1950): THERE IS AUTOMATIC SUMMATION FOR INDICES OCCURRING TWICE IF BOTH ABOVE AND BELOW.) In a Riemann space g_{ik} is positive definite; in a pseudo-Riemann one this does not hold. Then in a pseudo-Riemann space (as the spacetime around us) 0 distance is possible between points which are not neighbours. Then in such a space the metric generates a so called light-cone structure: the lines of 0 distance build up a kind of connection. Here we will not have pseudo-Riemann spaces, but still the problem of neighbours is to be discussed, since the metric tensor does not show the *topology*. (E.g., the same metric tensor is valid for a plane, a mantle of a cylinder and that of a cone.) This problem is relegated to Section 6, and until that we regard the topology as known.

So we have our selected infinitesimal triangle around x^i. Let us try with a *transformation*. Introduce a vector field $K^i(x)$, and shift all the points along this vector field in the manner

$$x^{i'} = x^i + \epsilon K^i(x) \tag{2.2}$$

with the *same* ϵ for all the points. Then, depending on the vector field, there are three possibilities:

1) The new triangle is geometrically *identical* with the original one, i.e., the three angles have remained unchanged, and so for the lengths of the sides. Then we call this shift a *Killing symmetry*.

2) The new triangle is *similar*: the angles are unchanged and the lengths proportional. Then the shift is a *conformal Killing symmetry*.

3) Neither is true. Then the shift is not a symmetry. (To be sure, weaker symmetries can be defined and sometimes are used (Katzin *et al.*, 1969). However somewhere one must draw a line, and we stop at conformal Killing symmetry.) In addition, for a symmetry we require that at the new points the matter remain the same (symmetry) or at least similar (conformal symmetry), i.e., that Φ_α show some scaling.

Now we are going to discuss the geometry in details: then some transformations are symmetries from geometric viewpoint, and later the consumer can check if Φ_α shows scaling or not.

We start with infinitesimal shifts, and this will be enough, because one can repeat the transformation in any times. So now $|\epsilon|$ is infinitesimal, and we can calculate up to first order in it. Note that

$$dx^{i'} = dx^i + \epsilon K^i{}_{,r}dx^r \tag{2.3}$$

where the comma stands for partial derivative. (The partial derivative generally destroys the definite tensorial structure in Riemann spaces, so formulae of relevance must be reformulable in terms of covariant derivatives, denoted by semicolon. For the definition of the latter derivative see the standard literature, e.g., (Eisenhardt, 1950). All our final formulae could be written into such a form.) Our previous conditions for a symmetry can be written as

$$ds'^2 \equiv g_{rs}(x')dx^{r'}dx^{s'} = e^{\epsilon\Omega}ds^2 \tag{2.4}$$

where $\Omega=0$ if the symmetry is not conformal.

Now, eqs. (2.2-4), in first order, lead to

$$g_{ir}K^r{}_{,k}+g_{kr}K^r{}_{,i}+g_{ik,r}K^r = \Omega g_{ik} \tag{2.5}$$

By means of covariant derivatives this equation can be rewritten as

$$K_{i;k} + K_{k;i} = \Omega g_{ik} \tag{2.5'}$$

which is the *Killing equation* (Eisenhardt, 1933); conformal if Ω differs from 0. In terms of Lie derivatives (Eisenhardt, 1933) the equation gets the even more compact form

$$L_K g_{ik} = \Omega g_{ik} \tag{2.5''}$$

expressing the fact that the change of the metric tensor is proportional to itself along the vector field **K** (0 if the symmetry is not conformal). This is just the reason for the similarity between the triangles.

If the metric of the space is given (which we assume henceforth), then eq. (2.5) is a system of linear partial differential equations, which can be solved somehow. Combinations with constant coefficients are again solutions, so the general solution is

$$K^i(x) = q^\tau K_\tau{}^i(x) \tag{2.6}$$

where the q's are constant; provided that we have got all the independent solutions $K_\tau{}^i(x)$. For conformal symmetries the corresponding Ω's add up in a similar manner.

For more than one Killing vectors the corresponding transformations form a group, since a sequence of symmetry transformations is a symmetry transformation by its result. Then one can calculate the commutator of two infinitesimal transformations by using eq. (2.3); the result reads as

$$[K_\alpha, K_\beta]^i = K_\alpha{}^r K_\beta{}^i{}_{,r} - K_\beta{}^r K_\alpha{}^i{}_{,r} \tag{2.7}$$

(WHERE ALL THE COMMAS COULD BE SUBSTITUTED BY SEMICOLONS AS WELL BECAUSE THE DIFFERENCE CANCELS DUE TO THE ANTISYMMETRIC COMBINATION). Since the commutator again would generate a symmetry transformation,

$$[K_\alpha, K_\beta]^i = c_{\alpha\beta}{}^\tau K_\tau \tag{2.8}$$

where the c's are the *structure constants* of the group.

3. CONTINUOUS SYMMETRY TRANSFORMATIONS IN THE 3 DIMENSIONAL EUCLIDEAN SPACE

Biological objects live around us in a 3 dimensional Euclidean space. There

$$g_{ik} = \delta_{ik} \tag{3.1}$$

This is a zero curvature space. For constant curvature the number of independent Killing vectors is ½$n(n+1)$; including conformal ones ½$(n+1)(n+2)$ (Eisenhardt, 1933). So in our case the number of possible independent symmetries is 10, of which 6 keep distances, while 4 are only similarities. The complete list is as follows:

$$P = \begin{pmatrix}1\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\\0\end{pmatrix} \qquad \begin{pmatrix}1\\0\\1\end{pmatrix}; \qquad \Omega_P = 0$$

$$J = \begin{pmatrix} 0 \\ +z \\ -y \end{pmatrix} \qquad \begin{pmatrix} -z \\ 0 \\ +x \end{pmatrix} \qquad \begin{pmatrix} +y \\ -x \\ 0 \end{pmatrix}; \qquad \Omega_J = 0$$

$$D = \begin{pmatrix} x \\ y \\ z \end{pmatrix}; \qquad \Omega_D = 2 \tag{3.2}$$

$$Q = \begin{pmatrix} x^2 - \frac{1}{2}r^2 \\ xy \\ xz \end{pmatrix} \qquad \begin{pmatrix} yx \\ y^2 - \frac{1}{2}r^2 \\ yz \end{pmatrix} \qquad \begin{pmatrix} zx \\ zy \\ z^2 - \frac{1}{2}r^2 \end{pmatrix}; \qquad \Omega_Q = x$$

with

$$r^2 \equiv x^2 + y^2 + z^2 \tag{3.3}$$

where boldfaces stand for 3-component entities colloquially called vectors of axial vectors. Generally **P** is called *translation*, **J** *rotation*, D *dilatation* and sometimes **Q** *inversion*, being in some connection with the inversions to the unit circle. The most general vector field for a symmetry transformation is built up from the above ones with 10 constant coefficients.

By evaluating the commutators (2.7) one gets

$$\begin{aligned}
[P_\alpha, P_\beta] &= 0 \\
[P_\alpha, J_\beta] &= \epsilon_{\alpha\beta\tau}\delta^{\tau\sigma}P_\sigma \\
[P_\alpha, D] &= P_\alpha \\
[P_\alpha, Q_\beta] &= \delta_{\alpha\beta}D + \epsilon_{\alpha\beta\tau}\,\delta^{\tau\sigma}J_\sigma \\
[J_\alpha, J_\beta] &= \epsilon_{\alpha\beta\tau}\delta^{\tau\sigma}J_\sigma \\
[J_\alpha, D] &= 0 \\
[J_\alpha, Q_\beta] &= \epsilon_{\alpha\beta\tau}\delta^{\tau\sigma}Q_\sigma \\
[D, Q_\alpha] &= Q_\alpha \\
[Q_\alpha, Q_\beta] &= 0
\end{aligned} \tag{3.4}$$

This is the *conformal group of 3-space*. Ignoring the conformal symmetries we get the symmetry group of the Euclidean space in stricter sense, $\{P, J\}$, which is the $E(3)$ group of 3 translations and 3 rotations.

Now we are ready with the widest possible group of continuous symmetry transformations for biological objects. Then comes the second condition that in the new points the matter be the same (or similar) as in the original ones. So a continuous conformal symmetry transformation will have the form (2.2), where K^i is as in (2.6), with the particular K's from (3.2), so that the material characteristics (density, colour, composition, etc.) show a scaling as well; a continuous strict symmetry will contain only $\boldsymbol{P}$ and $\boldsymbol{J}$ from (3.2) and the material characteristics will be strictly unchanged. (IT IS POSSIBLE TO BE MORE GENERAL. ONE MAY DEFINE TRANSFORMATION FOR THE MATERIAL DATA AS WELL, SAY CHANGES IN COLOUR, AND THEN A COMBINED SHIFT AND COLOUR CHANGE MAY BE A SYMMETRY. THIS CONSTRUCTION IS KNOWN IN SOME BRANCHES OF PHYSICS, AS E.G., THE SUPERSYMMETRY OF PARTICLE PHYSICS (GOL'FAND AND LIHTMAN, 1974) BUT THIS WILL NOT BE DONE HERE.) In what follows we assume that there is no problem to decide if the new point is materially identical with (or similar to) the original one.

Then by observation one can select the actually existing subgroup. The theoretical possibilities are limited because *generally* the commutator relations (2.7) cease to be closed if we omit symmetries from a group. To the end of this Section we ignore the *conformal* symmetries.

Now I try to list all the possibilities. (OF COURSE THE POSSIBLE SUBGROUPS OF E(3) ARE LISTED IN THE STANDARD LITERATURE (PETROV, 1966). HOWEVER, THAT IS NOT THE COMPLETE ANSWER FOR THE PRESENT QUESTION, AS WILL BE SEEN.) I hope that no case is forgotten, but if the Gentle Reader were to discover a missing one then he would be honoured. To begin with, observe that we have 3 independent 'directions' (say, $i = x, y, z$). Now, select, e.g., x. We have P_x and J_x in this direction. There are then 5 possibilities:

1) Both P_x and J_x are symmetries.

2) Only P_x is a symmetry.

3) Only J_x is a symmetry.

4) Neither P_x nor J_x is a symmetry, but a special combination of theirs $H_x = aP_x + bJ_x$ is.

5) No combination of P_x and J_x is a symmetry.

Here H_x is a helical symmetry, a combined rotation + translation along the rotational axis. Now let us start from the maximal symmetry downwards. Observe that the numbering 1, 2, 3 of the components is arbitrary.

6 symmetries:

$(\boldsymbol{P}, \boldsymbol{J})$. The matter is homogeneous and isotropic; no preferred point or direction.

5 symmetries:

ϕ. This class is empty, according to the Fubini lemma (Petrov, 1966).

4 symmetries:

$(\boldsymbol{P}, J_1)$. Homogeneity, with a preferred direction, which is an axis of rotation.

3 symmetries:

a) (P_1, P_2, P_3). Homogeneity without isotropy.

b) (P_1, P_2, J_3). Planar symmetry, with a rotational symmetry around an axis orthogonal to the planes.

c) (P_1, P_2, H_3). Planar symmetry, with helically rotated but otherwise identical planes.

d) (J_1, J_2, J_3). Spherical symmetry (rotations around 3 orthogonal axes).

e) (H_1, H_2, J_3). Rotational symmetry around the z axis with helical symmetries along any axis orthogonally intersecting the z one.

2 symmetries:

a) (P_1, P_2). Sequence of planes with homogeneity within each plane.

b) (P_1, J_1). Cylindrical symmetry.

1 symmetry:

a) (P_1). Translational symmetry in one direction.

b) (J_1). Rotational symmetry around one axis.

c) (H_1). Helical symmetry along one axis.

These are 12 different possibilities for non-conformal continuous symmetries of living organisms; in the cases when H's appear, there is a scale constant connecting translation and rotation. Including the 4 conformal symmetries the construction would go likewise, but this will not be done here.

4. ON THE DISCRETE SYMMETRIES

As mentioned earlier, the discrete symmetries in 3 dimensions are extensively listed in crystallographic literature. (See e.g., (Kittel, 1961) and further citations therein.) So we do not go into details, only classify the possible discrete symmetries into two classes. Namely

1) In addition to the $E(3)$ group the Euclidean space possesses 3 independent discrete symmetries, which can be chosen as reflexions to 3 orthogonal planes.

2) Any possible continuous symmetry, which is actually absent, can still appear at discrete steps. E.g., it is possible that P_1 is not a symmetry with arbitrary transformation parameter ϵ, but still it is a symmetry in steps $N\Delta$.

Again, a lot of combinations may exist. E.g. rotations and reflections to planes may be combined to reflections to sequences of rotated planes. Now, the discrete version of this is the familiar radial symmetry.

5. EXAMPLES

Up to now we restricted ourselves to the Cartesian coordinates x, y, z. Now, for explicit examples, it is useful to use other special coordinates, but we do not want to go into the details of coordinate transformations (for which see e.g., (Eisenhardt, 1950)). Only let us note that by performing the coordinate transformation $x^{i'} = x^{i'}(x^k)$ a vector is transformed as

$$K^{i'} = K^r(\delta x^{i'}/\delta x^r) \tag{5.1}$$

Two special coordinate systems will be mentioned. The first is the spherical polar system (r,θ,Φ):

$$\begin{aligned} x &= r\sin\theta\cos\Phi \\ y &= r\sin\theta\sin\Phi \\ z &= r\cos\theta \end{aligned} \tag{5.2}$$

with the corresponding inverses. The second is the cylindric system (ρ,φ,ζ):

$$\begin{aligned} x &= \rho\cos\varphi \\ y &= \rho\sin\varphi \\ z &= \zeta \end{aligned} \tag{5.3}$$

Now let us see some symmetries. First we return to the list of possible continuous strict symmetries.

3d, spherical symmetry.

By transforming all the 3 J's in (3.2) into spherical polar coordinates (not given here) one obtains that the transformations act on r = const. surfaces. Any point on such a surface can be rotated into any other. Therefore the symmetry exists if

$$\Phi_\alpha = \Phi_\alpha(r)$$

By other words, any material date can change only with the radial distance from a center.

3b, planar symmetry with rotation.

It is the most convenient now to use cylindrical coordinates. The rotation axis is taken in the φ direction. In the planes orthogonal to this axis there act two translations and one rotation, which is the $E(2)$ group. Such a plane is then maximally symmetric. Therefore nothing depends on the coordinates ρ, φ. Consequently the symmetry exists if

$$\Phi_\alpha = \Phi_\alpha(\zeta)$$

2b, cylindrical symmetry.

The symmetry direction is taken as z. Then, by transforming P_z and J_z one gets

$$P_z = \begin{pmatrix}0\\0\\1\end{pmatrix}; \qquad J_z = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

Consequently there remains

$$\Phi_\alpha = \Phi_\alpha(\rho)$$

1c, helical symmetry.

As mentioned, $H_1 = P_1 + qJ_1$. The symmetry direction is taken as z. Then in cylindric coordinates

$$H_1 = \begin{pmatrix}0\\q\\1\end{pmatrix}$$

The condition that a scalar obey the symmetry is, for the analogy of eq. (2.5")

$$L_K\Phi_\alpha = 0$$

which reads as

$$\Phi_{\alpha,r}K^r = 0$$

(Eisenhardt, 1950). Substituting the actual K^i and solving the partial differential equation, the result is

$$\Phi_\alpha = \Phi_\alpha(\rho, \varphi \;\; - \mathrm{q}\zeta)$$

Finally, consider a case when conformal Killing vector is also included. Let the *only* symmetry be

$$J_1 + qD$$

We take the symmetry direction z. Then in cylindric coordinates

$$(J_1{+}D)^i = \begin{pmatrix}\mathrm{q}\rho\\1\\\mathrm{q}\zeta\end{pmatrix}$$

Again, requiring this as a *strict* symmetry on the material fields, from the vanishing of the Lie derivative one obtains

$$\Phi_\alpha = \Phi_\alpha(\rho e^{-q\varphi},\ \zeta e^{-q\varphi})$$

By other words, in one 360° rotation the *equivalent* ρ and ζ values increase by a factor e^q. Consequently the structure is spirallic, with proportionally growing elements. *Similar* structures, but of course only with discrete symmetries, are well known (e.g., *Ammonites*). However, if this symmetry exists, chamber sizes, suture lines etc. must obey a very definite growth law with a single scaling constant. Such a scaling can and ought to be checked.

6. TOPOLOGY

As told earlier, the metric g_{ik} does *not* define the neighbours on a manifold. As an example, consider the mantle of a cylinder of radius ρ_0. Points ($\varphi = 2\pi$-ϵ, ζ_0) and ($\varphi = \epsilon$, ζ_0) are very near to each other if $\epsilon << 1$. On the other hand, performing the transformation (5.3) the two points get at a distance $(2\pi\text{-}2\epsilon)\rho_0$, because the mantle is cut just between the neighbouring points and they are rolled apart. So the coordinates and the metric are not enough; some extra information is needed about the possible "compactification" of the surface.

In the 3 dimensional Euclidean space we do have this information. However, possible configurations still are to be classified according to neighbourhood relations (connectivity). The problem has an extended literature, so here only a few simple examples will be mentioned; for the details cf. e.g., (Patterson, 1956).

Consider a compact *2 dimensional* entity (for such case the complete classification is known), e.g., the unit disc $\rho \leq 1$, $0 \leq \varphi < 2\pi$. This disc is *simply connected*. This term means that any closed curve on it can be reduced to any point or can be transformed into any other closed curve by *continuous* distortion (see the notion of homotopy groups (Patterson, 1956)). This is shown on Figure 1.

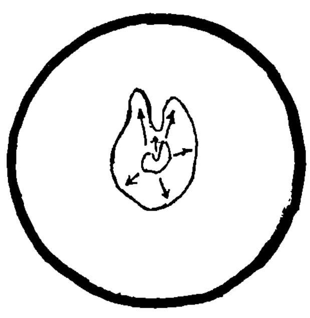

Figure 1

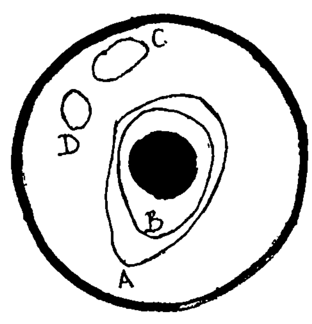

Figure 2

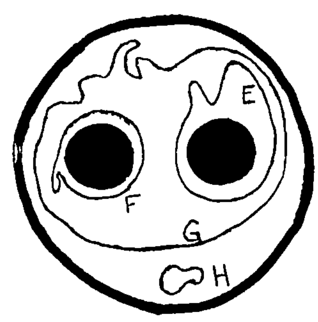

Figure 3

Now, punch a small hole around the center. The punctured disc is no more simply connected (Fig. 2.). The closed curves now are classified into 2 disjoint classes. Curves around the hole (say A and B) can be distorted into any other such curve but cannot be distorted into curves not surrounding the hole (*C* or *D*) and vice versa. In addition, a disc with a central hole can be distorted into another disc with eccentric hole. (THE DISTORTION, ON THE OTHER HAND, IS DETECTED BY THE METRIC STRUCTURE, BECAUSE DISTANCES BETWEEN POINT PAIRS ARE CHANGING. NEIGHBOURHOOD AND METRIC RELATIONS TOGETHER ARE ENOUGH FOR COMPLETE DESCRIPTION.) So for *topology* all the discs with one hole are equivalent.

Punch a second hole (Fig. 3). This disc you can deform into any other disc *with two holes*. Again, on it closed curves classify into four classes: surrounding the first hole (*E*), the second hole (*F*), both holes (*G*) or neither of them (*H*). And so on. These objects are inequivalent with each other.

Things go similarly for *bodies*. Consider a sphere. Each internal closed curve can be deformed into any other. This will not change if one digs a pit into it. However, driving a shaft completely through the interior is no more simply connected: curves surrounding the tunnel cannot be deformed into curves not surrounding it. At this point we deliberately stop.

It may obviously be important the possible ways of internal connections for the structure and operation of an organism.

7. CLOSING REMARK

This paper is by no means complete, and in addition at this point may seem rather pointless. However, it contains just the necessary mathematical background for the rest of the articles in the next issue. Our present purpose was not to enjoy pure mathematics but to help the understanding of some following articles.

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