Symmetry: Culture and Science
Vol. 1, No. 2, 1990, 119-138

SYMMETROSPECTIVE: A HISTORIC VIEW

Introductory essay to the Special Issue

THE KALEIDOSCOPE AND SYMMETRY (OR, A SYMMETROSCOPE)

PART 2: FROM SCIENCE TO ART (20TH CENTURY)

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In the first part of this paper we have seen that the kaleidoscope, contrary to the original intention of its discoverer, Sir David Brewster, had little success in the arts. More sophisticated devices gained, however, some importance in the mathematical and crystallographic works of Möbius and Fedorov, respectively. In this Part 2 we continue the history of the kaleidoscope in the 20th century which leads, in same sense, back to artistic problems. After surveying various new kinds of kaleidoscopes we provide a "periodic table" of all the possible devices formed by plane mirrors.

A NEW TURN OF THE KALEIDOSCOPE (MORE SYMMETRICALLY...)

After the "golden age" of geometric crystallography in the late 19th century, and the parallel kaleidoscope-related activities of E.S. Fedorov, as well as of Edmund Hess and others, this optical device disappeared from the scholarly interest. It did return, albeit at first metaphorically, in the 1930s. The book *Mathematical Kaleidoscope* by Hugo Steinhaus, a leading Polish mathematician, offered not only "colorful pictures" on the topic, but many interesting recreational questions and serious problems in connection with symmetrical figures, including regular tilings, models of polyhedra, shapes of crystals, close packings of equal balls, and so on. The original Polish edition *Kalejdoskop matematyczny* (Lwów, 1938) was followed by an English version in the same year, and later by a large number of other translations and enlarged editions. Interestingly, this book, which deals with various devices useful in mathematics, does not consider the kaleidoscope itself, even the very word disappeared from the English title: *Mathematical Snapshots* (New York, 1938, 1950, etc.). Many other translations, however, preserved the "kaleidoscope" in the title, including those in Russian, German, and Hungarian.

It was not necessary, however, to wait too long for the application of the kaleidoscope in forefront of mathematical literature. Notably, related topics appeared in two other pearls of the field, in W.W. Rouse Ball's book *Mathematical Recreations and Essays* (London, 1939), more precisely in a new chapter of its revised edition written by H.S.M. Coxeter (1939), and in Richard Courant and Herbert Robbins's book *What is Mathematics?* (London, 1941). In the first work, Coxeter — giving credit to Hess (1889) and also referring to Möbius — discusses those kaleidoscopes which we called cylindrical and trihedral. Moreover he also remarks on the possibility of further devices: three types of tetrahedra as well as certain triangular and rectangular prisms with internal reflecting surfaces. These kaleidoscopes, shaped by four, five, or six mirrors, respectively, lead to 3-dimensional solid tessellations. Later we will return to this topic. The book by Courant and Robbins (1941), which is not a "kaleidoscopic" work, but a systematic introduction to mathematics for both students and interested laymen, includes a short chapter on repeated reflections in cases very similar to the arrangement in the cylindrical kaleidoscopes, without referring to these instruments by name. They use a very simple, purely geometric approach focusing on rectangles and those kinds of triangles which are able to generate just by reflections non-overlapping coverings of the plane (we may call them mirror tilings or mirror tessellations). It would be interesting to go further with this approach and to show that there are only those four possibilities which are illustrated in the book (cf., the four types of cylindrical kaleidoscopes in Part 1 of this article, p. 34), but Courant and Robbins move to another interesting topic. They consider two circular mirrors, or two reflecting columns based on circles, illustrating the repeated inversion of circles (the inversion is a kind of angle-preserving transformation; here it is not associated with central symmetry which is the conventional meaning of the expression in geometric crystallography). We will deal, however, with simpler cases: in our discussion, except when indicated otherwise, the kaleidoscopes are formed by plane mirrors.

The real return of the kaleidoscope to geometric questions and its systematic application in this field came with the monographs by H.S.M. Coxeter (1948; 1961). Indeed, the Canadian mathematician not only revitalized synthetic geometry as a university subject, but also gave a new meaning to Brewster's old optical instrument. He analyzes the dihedral reflection groups using the kaleidoscope; moreover his results inspired an educational film *Dihedral Kaleidoscopes* (Coxeter, 1965). These devices are useful to present regular polygons and star-polygons by fixing appropriate line-segments in the object box. The possibility of kaleidoscopic representation of some regular arrangements in the plane, specifically rosette-, frieze-, and wallpaper-like patterns (symmetry groups), was also remarked by another geometer, the Hungarian László Fejes Tóth (1964). Considering analogous questions in 3-dimensional space, Coxeter discusses, independently of Fedorov, but referring to Möbius, the trihedral kaleidoscopes which can represent some regular polyhedra, or, in other words, Platonic solids. We have seen in Part 1 of this article (pp. 35-36) how to make such a kaleidoscope by three mirrors forming a "reflecting corner", and we considered all the possible types: there are infinitely many variable

(Part 1, Fig. 1) and three further types of trihedral kaleidoscopes. While the variable types are easy to imagine by standing a hinged dihedral (two-mirror) kaleidoscope on a third horizontal mirror, the form of the additional three types are more complicated. Coxeter (1961) describes an interesting device which shows the connections between these three types. He only uses rectangular mirrors, by means of which he is able to form just a part of the trihedral surface, not the entire corner, but still one can see the kaleidoscopic reflections if the position of the object is carefully chosen. The advantage of this partial kaleidoscope, whose surface has gaps at the corner and sides, is its variability: by rotating one of the mirrors we are able to simulate all of the above-mentioned three types of trihedral kaleidoscopes. Probably the easiest way to imagine this variable device, slightly modifying the original description, is to start with a cylindrical kaleidoscope of (90°, 60°, 30°), standing upright on a table, then to turn that face down — rotating around the edge on the table — which lies opposite to the angle of 60° (Fig. 3).

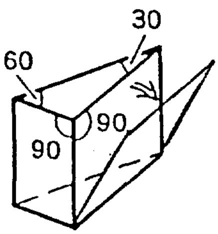

Figure 3: The partial (or "gappy") kaleidoscope: a device for simulating the reflecting surfaces of three trihedral kaleidoscopes. The idea of starting with the indicated cylindrical kaleidoscope is ours, while the main trick of rotating a mirror-face is due to Coxeter (1961, p. 280). Originally, he refers to a dihedral kaleidoscope with an angle of 60°, standing upright on a table, while the third mirror is in an appropriate horizontal position. The latter should be turned up, inversely to the direction indicated here. The present version of the device has a minor advantage: it makes clear the connection with the referred to cylindrical kaleidoscope.

The rotated face during the opening-out process reaches all those three positions where the three mirrors together, disregarding the gaps between them, represent the required trihedral kaleidoscopes. In those cases the device, with a suitable object, presents "spatial rosettes", namely icosahedral, octahedral and tetrahedral patterns, without ambiguity. Note that these three trihedral kaleidoscopes, considering in each case the entire surface without gaps, are not flexible: these three cannot be transformed into each other with simple rotations.

Later Coxeter went on to develop the most exciting type of the trihedral kaleidoscopes as a useful educational tool for demonstrating a set of polyhedra. It is the first one reached when opening out the cylindrical kaleidoscope (Fig. 3), and the last one in our earlier list (Part 1, p. 35). An interesting property of this type is that the sum of the three angles of the faces at the corner (the rounded data are 37°, 32°, 21°) gives a right angle. This means that the surface of the instrument can be obtained from a square by simple paper-folding, with the three inner faces lined with a reflective tape, or replaced by mirrors. In a version of this instrument we may make a small hole at the corner (or vertex) of the three mirrors. In this case, the device can be turned in the direction of the light as with the classical kaleidoscope. Coxeter does not use liquid as an object in the kaleidoscope, which was Fedorov's suggestion, but his method of construction shows some special line-segments on the original square. Marking these lines, by inserting thin strips at the corresponding place on the mirrors, we may see the image of an icosahedron and a dodecahedron, as well as some star-polyhedra (Fig. 4). The typical activity with this kaleidoscope is not the simple turning of the device, as in the case of Brewster's original one, but the fixing of the appropriate line-segments. Coxeter's foldable kaleidoscope really requires some geometrical creativity from the user, and it may contribute to a better understanding of the theory of polyhedra. A very accurate version of this trihedral kaleidoscope was completed during the above-mentioned educational film project in the mid 1960s. Unfortunately the second film about this advanced kaleidoscope was never completed, but the mathematical discussion of this instrument is available in a later monograph by Coxeter (1974). Incidentally, Coxeter speaks about icosahedral or polyhedral kaleidoscopes, emphasizing the presented image, not the actual trihedral surface of the instrument. We prefer to use the term trihedral kaleidoscope; the advantage of this will become clear later. Another useful name was coined by the British crystallographer Alan Mackay (1967) who speaks about "corner kaleidoscopes". He uses these devices not only for demonstrating various polyhedra, but also for presenting models for polyhedral viruses. Indeed, inserting spheres we may present icosahedral packings and other related structures.

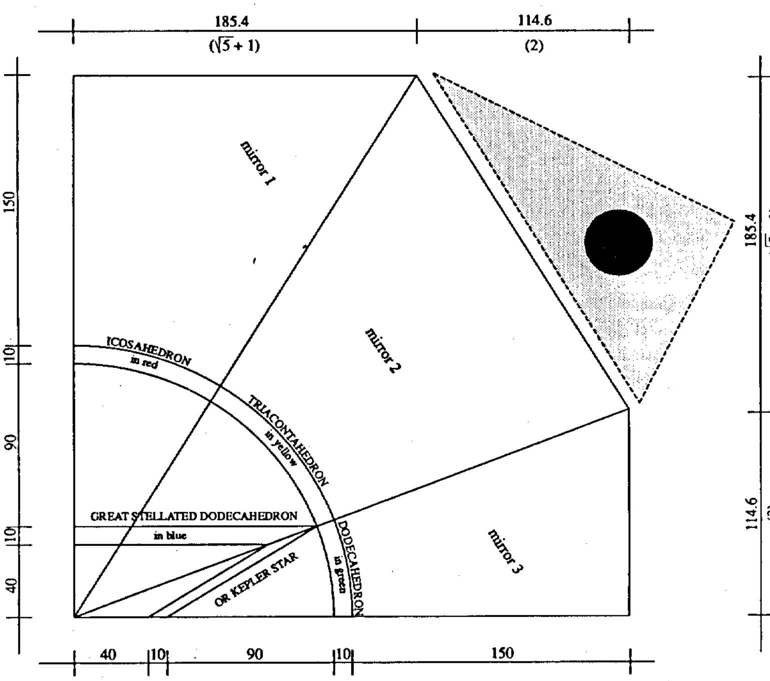

Figure 4: The foldable kaleidoscope of Coxeter was also rediscovered by Nikolaus and Caspar Schwabe. It is given here in their layout: starting with a given square one may easily form the surface of the corresponding kaleidoscope by simple folding, while the fourth triangle (top right) should be cut and reoriented to close the trihedral surface with the exception of an eyehole for viewing (black circle). The Schwabes mark, except in one case, arcs — instead of line-segments — by which they may present the images of spherical models of the identified polyhedra. Moreover the line-segments and arcs have different colors, thus one may see the colored images together. Originally all the data are in millimeters. Note that both the top and the right sides of the square are divided, as the data in parentheses show, according to the ratio of the golden section. Caspar Schwabe calls it the "golden section kaleidoscope" (copyright, 1986). In the case of the earlier version of Coxeter (1974, p. 24) the smallest mirror, which is marked here as No. 3, is in the middle, and thus his layout does not make possible to utilize the fourth corner for closing the trihedral surface. Coxeter describes the possibility of shaping the image of some further polyhedra by inserting appropriate line-segments: the small stellated dodecahedron, the great dodecahedron, and the icosidodecahedron. (The layout given here is made following a sketch by Caspar Schwabe; the computer drawing is courtesy of Mihály Szoboszlai.)

While Coxeter started in the Brewsterian tradition, A.V. Shubnikov and V.A. Koptsik (1972) continued the Fedorovian one. Although the new interest of Russian crystallographers in kaleidoscopes was indicated much earlier when Shubnikov (1939) published a paper on the subject and also discussed this topic in a comprehensive monograph on symmetry (1940), the international recognition of these results had to wait until the publishing of the English version of the cited book *Symmetry in Science and Art*, written jointly with Koptsik. The authors discuss kaleidoscopes not in a separate chapter, but ever more advanced versions appear in parallel with the generalization of the concept of symmetry. This "invariant" presence of kaleidoscopes in the corresponding chapters demonstrates clearly that the instrument is not only a toy, but also a useful tool in education. The original Russian edition of 1972 (p. 32), not Koptsik's revised English version, makes the same mistake about the classical dihedral kaleidoscopes that was in the center of the Kircher-Brewster controversy (the allowed angles are 360°/*n*, instead of restricting them to 180°/*n*). This fact supports the view that the two traditions were really independent. Shubnikov and Koptsik survey, however, not only the classical kaleidoscopes, but they go far beyond.

Discussing the symmetries of friezes, linear ornaments on one side of a band, Shubnikov and Koptsik describe those two kaleidoscopes which present such patterns (cf., also Fejes Tóth, 1964, p. 41): the first one has two parallel mirrors facing each other (a special case of the dihedral kaleidoscope; see Part 1 of this article, p. 34); while the second one has three mirrors where the former arrangement of two parallel mirrors is extended by a third perpendicular one giving a U-shape (all the mirrors facing inwards of the U). We should add that the latter arrangement can be interpreted as a special case of the infinite class of trihedral kaleidoscopes (see Fig. 1 in Part 1 of this article). Namely, if n is infinitely large, then the angle of 180°/n between the corresponding two mirrors is 0°, so that they are parallel. The common property of both special cases, the dihedral one discussed earlier and the new trihedral one, is that they present frieze patterns. We may also obtain these two special kaleidoscopes by removing either two opposite mirrors or just one, respectively, from a cylindrical kaleidoscope of rectangular cross-section.

It has been mentioned that the trihedral kaleidoscopes, Shubnikov and Koptsik call them Fedorov kaleidoscopes, are able to represent local arrangements of crystallographic structures, namely, symmetrical systems of points around a singular lattice-point (point groups). A further step in the book is the discussion of those kaleidoscopes that are able to represent global crystallographic structures, specifically 3-dimensional periodic systems (space groups), more precisely a part of them (cf., Coxeter, 1939, pp. 159-160). These instruments are based on the internal surface of various polyhedra, such as a rectangular prism, for example. The function of this instrument can be easily understood by the analogy of a mirror hall, where we, as an object, are standing in a closed space surrounded by mirrors (suppose that the floor and the ceiling are also covered by mirrors). We need a mirror hall where we may occupy any position inside without causing an ambiguity of images. Although mirror halls were popular in earlier periods, in the most cases the spectators could enjoy the phenomenon only from a few marked spots (cf., Greenslade, 1982; Walker, 1986). Now we require total freedom in selecting our viewing position. In the case of a mirror hall formed as a rectangular prism we have around us four mirrored walls as in the cylindrical kaleidoscope of rectangular cross-section. This part of the mirrors presents a periodic pattern of our images as standing on an extended floor. Two additional mirrors at the top and the bottom multiply the horizontal layer of our images to further layers in both directions up and down. Finally, we see a finite part of an infinite 3-dimensional pattern.

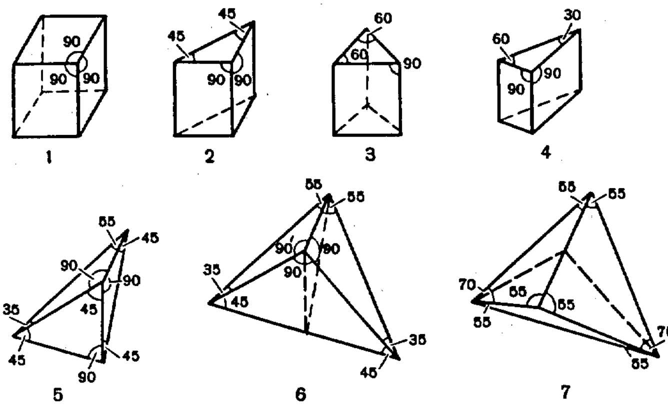

Figure 5: Shubnikov and Koptsik's polyhedral kaleidoscopes. The first four with prismatic shapes can be obtained from the four types of cylindrical kaleidoscopes by closing them at both open ends (adding a reflecting top and a similar bottom). The other three have tetrahedral shapes. The first tetrahedron (type 5) can be derived by cutting a cube into 48 congruent parts along all the symmetry planes; the second one (type 6) can be constructed by joining the earlier tetrahedron and its mirror image as the dashed lines show; finally the last one (type 7) can be obtained in a similar way by joining two tetrahedra of type 6. Each of these seven polyhedral kaleidoscopes should have a dihedral kaleidoscope at each of the edges, and a trihedral one at each of the vertices, always with the allowed angles which were discussed earlier in Part 1 of this article, pp. 33 and 35. Note that 35°, 55°, and 70° are rounded. (After Shubnikov, 1940; Shubnikov and Koptsik, 1972, Fig. 188, with a minor modification.)

Shubnikov and Koptsik find that there are altogether seven kaleidoscopes for presenting spatial patterns (Fig. 5). The proof of the completeness of their list is not given, but all of the types are described in detail. We suggest naming them, according to the arrangement of the mirrors, polyhedral kaleidoscopes (not in Coxeter's sense, but consistent with the terminology of dihedral, cylindrical, and trihedral kaleidoscopes). The polyhedral kaleidoscopes are surprising from many points of view. The classical literature on the subject often emphasized that it is impossible to make a kaleidoscope with more than four mirrors. In the case of this new kind of kaleidoscopes the cited statement is not true: one type has six mirrors (rectangular prisms), three other types have five mirrors each (triangular prisms). We are facing, however, new technical problems: the mirrors form a closed polyhedron, therefore it is not obvious how to see the images from outside, and how to illuminate the object (supposing that it is not a mirror hall, but a small polyhedral kaleidoscope). The first problem can be solved easily by drilling a small eyehole on the polyhedral surface. The second one is more complicated, but Shubnikov and Koptsik have an interesting suggestion. By using an electric bulb or a flashlight as an object inside the kaleidoscope, the problem of illumination will be also solved. Of course this method strongly limits the image; we will always obtain just a set of illuminated points. We have, however, another suggestion: to make small holes at the vertices, which will not disturb the reflections too much, and to illuminate via these holes an arbitrary object placed inside the kaleidoscope. We may use one of the holes at the corners as an eyehole. A further possibility is to form the surface of the kaleidoscope from one-way mirrored glass, like that one used in some sunglasses, which makes the illumination easy. It would be interesting to combine the polyhedral kaleidoscopes with various light effects, including laser illumination, or even making holographic objects inside. We may also project objects from outside into the kaleidoscope by classical optical methods.

Sophisticated systems of mirrors are also crucial for the recent attempt at building optical computers operated by laser light. The polyhedral kaleidoscopes — it is just a fuzzy idea — could be useful in constructing optical memory, that is, to keep some visual information inside the closed space of the device, then to retrieve it by opening a shutter-like hole. It also gives a simple tool to demonstrate the "swallowing up" of light by black holes, now a favorite topic in astrophysics and cosmology, where the light is not lost, but kept inside. Devices with curved mirrors may demonstrate curved spaces and their symmetry operations which are important in the theory of relativity and astrophysics.

To make our list of kaleidoscopes formed by plane mirrors complete, we should add some additional kinds that are not mentioned by Shubnikov and Koptsik. Consider those that are between the kaleidoscopes providing 2-dimensional and 3-dimensional patterns (assuming that we have a flat enough object, thus the dimensionality of the observed pattern is determined by the reflections and not by the shape of the original object). In some sense, we are moving towards a two-and-half-dimensional case. Adding a reflecting bottom to each of the four cylindrical kaleidoscopes (it is the same as removing the top of each of the four prismatic kaleidoscopes in Fig. 5) we obtain kaleidoscopes in the shape of an open box. Its cylindrical part presents a wallpaper pattern, while the bottom will double the whole image. Similarly we may add to the U-shaped kaleidoscope a perpendicular mirror at the back (a box without the front and top), where the new mirror doubles the frieze. It is not necessary to deal with the doubling of rosettes, because the variable trihedral kaleidoscopes (see Fig. 1 in Part 1) do precisely that. We may add, however, a horizontal top to these types which would ultimately give us a hinged dihedral device between two parallel mirrors (trough-like shape). Each of the dihedral kaleidoscopes presents a rosette, while the two parallel mirrors, as in the case of frieze patterns discussed earlier, line up copies of this rosette, giving finally a pattern similar to pearls on a stretched string or carvings on a rod. It is easy to see that we have obtained by this derivation all the possible types of kaleidoscopes of the corresponding kinds. The patterns discussed here (two-sided rosettes, two-sided friezes, two-sided wallpapers, and rods) are also considered in mathematical crystallography. For example, the two-sided wallpaper patterns are useful when studying various layers, including those of liquid crystals, thin films, membranes, and biological tissues. It is interesting to note that the terms frieze and wallpaper refer, similarly to the everyday meaning, to such ornamentations that are available only on one side of a surface, while the back is fixed to the wall. Now we have spatial generalizations of those planar patterns where the ornamentation is visible from both sides as, for example, in a fence or stained glass. Among textiles we find examples of both one-sided and two-sided designs.

And what about the presentation of colored symmetries, where the equivalent elements change their colors periodically as in the case of a chessboard? Of course the mirrors of a kaleidoscope cannot alternate the colors of the images. Although we may start with a colored object — the commercial dihedral kaleidoscope uses pieces of colored glass or beads — we obtain just symmetric patterns with colors, not colored symmetries. In other words, the repeated images, if we disregard the reduced brightness, have exactly the same color composition, and there is no periodic change of colors. The coloring of some mirrors, however, may help a bit in the cases of those devices which represent two-sided rosettes, friezes, and wallpapers. The two-sided patterns are analogous to their one-sided, but two-colored versions. This statement is not obvious, but one can imagine that the new opportunity of reversing the pattern to the other side is equivalent to using a second color on the same side. More precisely, we should accept a generalized understanding of the two-colored patterns to make this statement valid. The 17 types of one-sided wallpaper patterns (groups), discussed very briefly in Part 1 of this paper (pp. 34-35), can be generalized to 80 types of two-sided wallpaper patterns (this set also includes the 17 one-sided cases as those where we do not make use of the bottom side). These 80 types were first enumerated by Carl Hermann in 1929 and, slightly later in the same year, by Leonhard Weber. The latter author illustrated all of them on one side of the plane by projecting the patterns from the bottom side to the top, and marking them black. Specifically, Weber's one-sided patterns consist equal black and white triangles that represent, respectively, the original arrangements on the bottom and the top of the two-sided plane. This step clearly belongs to the preliminaries of the theory of two-colored (black-and-white) patterns, which are considered on one side of the plane. Moreover, it could have inspired Heinrich Heesch, who first proposed this approach (later Shubnikov, independently from Heesch, also introduced the concept of black-and-white symmetries). Indeed, the 80 patterns of Weber consist of 46 truly two-colored types of wallpaper patterns, where there are separate black and white triangles; 17 gray types, where the bottom and the top patterns were originally mirror images, therefore the projection led to coinciding black and white triangles; and the 17 one-colored types, where the bottom side was not utilized, therefore there is no second color. The latter two sets of 17 gray types and 17 one-colored types are geometrically the same. The doubling of this set as gray and one-colored types has, however, an appropriate interpretation: if the black and white colors represent two different physical properties (e.g., negative and positive magnetism), the gray types are neutral, while the one-colored types are associated with one of the properties. Thus, the crystallographers usually speak about 80 types of black-and-white wallpaper patterns, including the degenerate gray and one-colored cases (46 + 17 + 17 = 80). These 80 types really correspond to the 80 types of two-sided, but one-colored wallpaper patterns. We saw earlier in Part 1 (p. 34) that exactly four of the 17 one-sided wallpaper patterns can be represented by kaleidoscopes, specifically by cylindrical devices. If we extend a cylindrical kaleidoscope by a colored bottom mirror, we see both the original wallpaper pattern and its colored mirror image. Even the "projecting" of the colored figures to the other side can be simulated by standing a flat object on the colored mirror (suppose that it has no thick glass part, that is, the object is lying directly on the reflecting surface). In this case, however, the original object and its mirror image would coincide, therefore we cannot have truly two-colored patterns, only gray ones. Indeed, using a flat transparent object and a gray colored mirror, we may turn the object gray (as the result of the mixture of the colors of the transparent object and of the coinciding gray image), and all the repeated images would have the same mixed gray color. In this way we are able to represent four types of the gray wallpaper patterns: the gray versions of the previously discussed four one-colored patterns. This means that we may present altogether eight degenerate types of the total 80 types of two-colored (black-and-white) wallpaper patterns: four gray and four one-colored types. Similarly, we may represent four degenerate types — two gray and two one-colored — of the total 31 types of two-colored frieze patterns (17 truly two-colored, 7 gray, and 7 one-colored). Interestingly, when we extend the kaleidoscope made of two parallel mirrors by a third one to form a U-shaped arrangement we have two choices: to use a usual uncolored mirror to present another type of the 7 one-colored frieze patterns, or to use a colored mirror to present the gray version of the original frieze pattern. In the case of two-colored rosettes, again we may represent only degenerate cases, moreover all of them: the infinitely many gray ones by the variable class of trihedral kaleidoscopes, discussed in Part 1 (Fig. 1), with a colored horizontal mirror, and the infinitely many one-colored ones using simple dihedral kaleidoscopes. Note that all of the represented cases are almost trivial, thus the kaleidoscopes could not help us to go deeper into the theory of colored symmetries. Probably the idea of coloring the mirrors, as well as of using various illuminations, may have more importance for aesthetical experiments than for the study of crystallographic symmetries. There is, however, a rather artificial possibility for representing real colored symmetries: we could start with a colored symmetric (i.e., symmetrically colored) object and fit it into a kaleidoscope, if it is possible, in such a way that produces the periodic repetition of colors. For example, if we cut the usual black-and-white chessboard into four triangular parts along the diagonals, we may reconstruct the whole board again using just one quarter and a dihedral kaleidoscope with the angle of 90°. The similar diagonal cuts are always suitable for similar purposes if the chessboard has $n \times n$ squares, where n is an arbitrary integer, greater than 1. Moreover, in the case of odd numbers we may cut the chessboard into four squares along the median lines (this does not work, however, for even numbers). Strictly speaking, the kaleidoscope does not make colored symmetric patterns, only continues an initial one. Of course, not all patterns are suitable for similar kaleidoscopic continuation even if we select special positions for them. This remark may raise further questions about the classification of colored symmetric patterns.

Of course the group theoretic equivalent of all the kaleidoscopes, including the listed new ones, are well known in mathematics, not as optical devices, but as symmetry groups generated just by reflections. Coxeter (1948) gave a survey of the group theoretic aspects of the topic, including the n-dimensional analogues ("generalized kaleidoscopes"), when dealing with regular polyhedra and politopes. The pioneers of the geometrical theory of groups include Dyck, Klein, Poincaré and Schwarz, who were very active in this field in the late 19th century. We should also mention the name of Edmund Hess (1879; 1882; 1883; 1889) who, contributing to the theory of spherical tilings and the related polyhedra, rediscovered Möbius's trihedral kaleidoscopes. He worked independently of Fedorov (1883; 1885; 1890; 1891), therefore we should give more credit to him in connection with the trihedral devices (Möbius-Hess-Fedorov kaleidoscopes). The concept of reflection gained new importance in the approach of Friedrich Bachmann (1959) who proposed to use this to build the foundations of geometry. Interestingly, referring to a group theoretic paper by Vinberg (1981), there are no crystallographic reflection groups, as well as no generalized polyhedral kaleidoscopes, in non-Euclidean hyperbolic (or Bolyai-Lobachevskii) spaces of large dimension (cf., Stewart, 1986). The theory of black-and-white and multicolored symmetry, pioneered by Heesch's paper in

1929, Shubnikov's monograph in 1952, and Belov's coauthored papers in the mid 1950s, became a dynamic field on the border of mathematics and crystallography which led to some individual monographs (Jawson and Rose, Loeb, Koptsik, MacGillavry, Wieting, Zamorzaev, Zamorzaev et al.) and many important papers (Coxeter, Donnay, Schattschneider, Senechal, Schwarzenberger, van der Waerden and Burckhardt, and others). Interestingly, but perhaps not surprisingly, the topic of colored symmetry was also considered in the arts, not only on the level of intuitive construction of related patterns, but also in the systematic classification of them. Specifically, the Dutch graphic artist M.C. Escher and the British textile engineer H.J. Woods, who published a series of papers in the mid 1930s, significantly contributed to the field. The graphics by Escher became widely used for illustrating colored symmetries in the mathematical-crystallographic literature, while the approach of Woods was less known — with the rare exception of the monograph by Shubnikov (1940) — until the rediscovery by the crystallographer Donnay, and more recently by the mathematician Crowe and the anthropologist Washburn.

THE MYTH OF THE KALEIDOSCOPE QUICKENS

There were many mathematical congresses in the early 1980s where the participants twisted Rubik cubes during intermissions, but there were also geometrical conferences in the 1970s where the mathematicians handed over to one another Coxeter kaleidoscopes. Moreover, the interest in kaleidoscopes also reached the circles of artists and designers, making the myth of such applications real. A group of students around Keith Critchlow, who authored artistic-geometrical monographs on polyhedra and patterns, also became interested in kaleidoscopes, focussing on regular polyhedra (these kaleidoscopes were available in the Rudolf Steiner Bookstore, behind the British Museum in London). Later an architect designed a more advanced version of such devices for presenting icosahedral structures. It is marketed as *KaleidoSphere* by a company of the same name (Wilkes-Barre, Pennsylvania). After examining the polyhedral kaleidoscopes for presenting 3-dimensional symmetries, our fired-up imagination may suggest further steps. It would be interesting to connect internal mirroring surfaces of cubes, following the well-known models of the hypercube, to visualize 4-dimensional symmetry groups, or at least to present associated images. Note that the topic of 4-dimensional polyhedra gained some importance in various movements of abstract art. One of the leaders of the cubist movement, Marcel Duchamp, even made experiments with mirrors in connection with his interest in 4-dimensional space. Moreover he mentions in his notes a "3-sided mirror" (cf., Henderson, 1983, p. 154).

The advanced kaleidoscopes can probably be applied in visual psychology, where both the use of mirrors and of group theory have some tradition. Thus, mirror tracing — following a star-polygon with a pencil when the figure cannot be seen directly, but only in a mirror — is a well-known experiment in psychology from the 1910s. The concept of groups was introduced into the theory of perception relatively early by Ernst Cassirer's paper of 1938, while the school of Jean Piaget adapted it as a reference for analyzing the structure of cognition in developmental psychology. In the 1960s William Hoffman suggested describing various perceptual invariances by Lie transformation groups. Kaleidoscopes, connecting visual effects and symmetry groups, may help the studies in these directions, or at least may provide an additional domain in experimental psychology. For example, it is interesting to consider the orientation of individuals not only in a maze, but also in mirror halls or their smaller kaleidoscopic versions. The left-right orientation can also be investigated on a broad scale. Moreover, by applying polyhedral kaleidoscopes we may leave the "flatland" — using the title of Edwin Abbott's book — of everyday life, where we can rarely see spatial structures. (Even in the case of skyscrapers the outside view of the walls hides the structure of the building; while being inside we either recognize the "flatlands" in various levels, or move between them just seeing the elevator shaft.) The observations in the "mirror world" may also have some didactic advantage. Thus the use of mirrors for illustrating left and right pairs of particles, molecules, and biological shapes became customary in popular literature. Lewis Carroll's novel *Through the Looking-Glass and What Alice Found There* in 1872 gave not only an artistic interpretation of a mirror world, but the mathematician-author anticipated some later developments in science (this work also contributed to the preservation of the expression "looking-glass", which was earlier a much more popular alternative to mirror than today). Robert Routledge (1891), in his survey on the discoveries and inventions of the 19th century, discusses the contrast between the mirrors of the kaleidoscope, producing harmlessly symmetrical images, and of the *polemoscope* (from the Greek *polemos*, war), a periscope like device serving the artillery to observe the effect of their shots. Note that systems of mirrors are often used to display objects or events from many points of view concurrently, as in the case of security systems of supermarkets, some operating theaters, and museums (see also the use of various optical devices including telescopes, periscopes, prisms, and more recently optical fibres in many fields). Similar methods also gained some importance in arts. For example, many photo artists have made experiments with kaleidoscopic views of everyday objects. Some video clips apply kaleidoscopic effects, which is obviously the cheapest way to have a *corps de ballet* by multiplying a single performing dancer and automatically "coordinating" the parallel or mirror motions of the "group". The most practical applications of systems of mirrors, or mirroring surfaces, however require the opposite "anti-kaleidoscopic" principle: the goal is not the multiplication of images, but the exclusion of the kaleidoscopic effects which could disturb the observation or orientation. Thus the mirrors in a dance studio serve the checking and correction of gestures in the 3-dimensional space individually, without any desire to multiply the dancer. Similarly, it is better if the surface of a revolving door are nonreflecting; otherwise we may find ourselves in a rotating kaleidoscope. The problem of non-reversing single mirrors, which can be solved using curved mirrors, goes back to Plato, but it has gained a new importance for contemporary inventors (cf., for example, David Emil Thomas's U.S. Patent No. 4,116,540 in 1978). The same problem can also be solved by a dihedral arrangement of plane mirrors, which may present the mirror image of the mirror image, that is the original view of an object.

We discussed earlier, very briefly, the use of circular mirrors, or reflecting columns, in the book by Courant and Robbins (1941). This topic received a new emphasis when the British mathematician Caroline Series (1990a; 1990b) connected this topic with non-Euclidean geometry and the theory of fractals and chaos. The repeated reflections in circular mirrors — Series speaks about non-Euclidean kaleidoscopes — produce smaller and smaller circles and we may present in this way various non-Euclidean patterns. (Indeed, this system is able to produce, by successive inversions of circles, congruences of non-Euclidean hyperbolic planes). Moreover, the ever smaller circles lead to some questions on the geometry of fractals. Note that Helmholtz was probably the first scholar who suggested that convex mirrors offer an image of non-Euclidean space, moreover this idea could also have inspired Poincaré to construct his models of non-Euclidean geometry (cf., Henderson, 1983, p. 154). It is interesting to add that reflecting circular cylinders are also used in anamorphic art (from the Greek *ana*, again, and *morphê*, shape), an idea going back to Leonardo and other artists in the Renaissance. The distorted and hardly recognizable picture can be seen correctly only by seeing its reflection in an appropriate cylinder stood perpendicularly (cf., Gardner, 1975; Leeman et al., 1976; Moscovich, 1988). Interestingly, this general idea, considered mostly as an artistic curiosity, received a serious technical application when the Twentieth Century Fox used anamorphic lenses to produce the wide screen motion picture *The Robe* (1953). The lenses projected the wide image onto the standard film, which was projected, again by anamorphic lenses, to a wide screen. In addition to this, the traditional form of anamorphic art with cylindrical mirrors also survived, and it was reintroduced to the galleries of modern painting. For example, a Hungarian graphic artist, István Orosz, uses this idea to present visual ambiguity: in his case a recognizable picture is the input, not a meaningless distorted figure, which is transformed by the cylindrical mirror into another picture. Turning back to special mirrors, we should mention the artistic-photographic experiments with various curved mirrors made by some members of the *Bauhaus* movement in the 1920s, especially by László Moholy-Nagy (cf., his monograph *Malerei, Fotographie, Film*, München, 1925, which is translated as *Painting, Photography, Film*, London, 1968). Mirroring effects were also used in the kinetic sculptures and light mobiles, pioneered by the same group of artists. Escher's lithographs *Still Life with Reflecting Sphere* (1934) and *Hand with Reflecting Sphere* (1935) demonstrate the interest of another artist in related questions. Curved mirrors are also used for entertainment in various fun houses and science exhibitions, where we may see our transformed image. The behavior of animals and small children in the front of a plane mirror, or a system of mirrors, is often studied in ethology and developmental psychology. (It is used in experiments testing the recognition of the self). The mirror even became a metaphor in the title of a book about the natural history of human knowledge, written by Konrad Lorenz, one of the pioneers of ethology (*Die Rückseite des Spiegels*, München, 1973; English translation, *Behind the Mirror*, London, 1978). The idea of "mirroring" is also well-known in art therapy, especially in those cases when the clients miss someone. This relatively simple method provided good results in the case of children who were separated from their mothers: the client is asked to draw an abstract image while the therapist simultaneously makes the mirror image shaping together a nonseparable symmetric picture. It is also interesting to observe that phenomena associated with mirrors often feature in children's adventure novels. For example, the frequent motif of a mysterious car in the night whose light suddenly appears and disappears opposite to the young heroes' car, and which is produced by a simple mirror at a curve, can be interpreted as an exercise for the recognition of the self in a more complicated situation. Mostly the heroes, after some exiting pages, solve this problem, realizing that they are frightened by the mirror image of their own car. Illusionists also use various mirror systems. The "ghost illusion", especially the method patented by Pepper and Dricks, was a very popular trick in late 19th century (Routledge, 1891). The more complex systems of mirrors and their kaleidoscopic effects in mirror halls are also impressive attractions in some scientific and educational exhibitions.

The kaleidoscope is also useful in explaining the composition principles of Bach or Schoenberg. Both of them desired to use not only the root position of a basic theme, but also its symmetric images: *rectus* (or crab), *inversus* (mirror), and *recto-inversus* (crab-mirror). All of these symmetric variations can be seen using a dihedral kaleidoscope with the angle of 90°. It should be stood perpendicularly on the scoresheet of the basic theme, preferably in such a way that one of the mirrors is parallel to the lines, while the other one is perpendicular to them. In the latter mirror we see the *rectus* of the theme, in the parallel one the *inversus*, finally, the fourth pattern in the virtual mirrors would give the *recto-inversus*. Of course, it is one thing to see the scores of a basic theme and its symmetric variations thus derived and another to actually listen to the music. While the described symmetry transformations are almost obvious by visual perception (symmetries in plane), the most people will not realize them using just acoustic perception (symmetries in time). The physicist-philosopher Ernst Mach delivered a popular scientific lecture in 1871 in which he discussed and performed the "mirror music" (this German paper is also available in English: "On symmetry", see in his book *Popular Scientific Lectures*, La Salle, Ill., 1893; reprint, 1986, 89-106). Similar mirrorings also appear in language games and experimental poetry. Such a verse even inspired the composer Webern. The history of palindromes (reversible words or sentences) goes back to ancient times; often these were associated with magical meanings. It is not surprising that mirrors are also useful in children's interdisciplinary music education, where the basics are taught not only acoustically, but visually as well. Moreover, sometimes music, poetry, drawing, and geometry classes can be conducted together, as some Hungarian programs impressively demonstrated (Budapest, Kecskemét, Pécs). In the German-speaking countries Hugo Kükelhaus was a promoter of the complex use of the sense organs in education. He organized various travelling exhibitions, where the kaleidoscope also featured as an educational tool (see, e.g., Kükelhaus, 1975; cf., Baltrusaitis, 1986; Arn, 1990).

It has become clear that even the geometrical-optical aspect of the simple kaleidoscope is not a totally closed chapter. For example, to calculate in general the number of images of an object between two plane mirrors at an arbitrary angle is not as easy as it has been believed. An-Ti Chai (1971) found that some textbooks give erroneous formula even in simple special cases. Chai's short paper reconsidered the problem and gave a correct formula. If two mirrors are hinged, as in the case of the polyangular dihedral kaleidoscope, we may use this device as a teaching aid or a tool in geometrical construction. A paper by Jack Robertson (1986) investigates this possibility, replacing the ruler and the compass of the classical Euclidean construction by this instrument. There is relevant literature in mathematical education on how to use mirrors (plastic ones are available!) in order to understand the geometric properties of various figures; see *The Mirror Puzzle Book* by Marion Walter (1985) and many other publications and educational kits on *Mira*, *Kaleidoscope Maths* and related topics. Dale Seymour Publications in California and Tarquin Publications in England have fine selections about this approach. Related questions are often discussed by various journals on mathematical education. Returning to optics, Thomas Greenslade (1982) described how it is possible to use multiple mirror images in present-day physics education. Interestingly, most of his illustrations are adopted from 19th century textbooks. In another case the observation of a child led a physicist-father to investigate an attractive phenomenon, and finally to publish a joint paper about the reflection in a polished tube (Laurence Marshall and Emma Marshall, 1983). This device can be considered to be a cylindrical kaleidoscope with circular cross-section, which is formed by infinitely many plane mirrors (here we use the word cylinder in the everyday sense). True, it presents patterns without ambiguity only in special cases. The authors describe that case where the object is a central spot (it is a central hole in a covered end); here we see a set of concentric rings. Jearl Walker (1985) in his series in the *Scientific American* reconsidered the classical kaleidoscopes, and raised the question of those we call cylindrical kaleidoscopes with polygonal cross-sections. The readers had one month to think about the problem, after which Walker supplied the solution. Obviously the author very carefully studied the modern literature on kaleidoscopes and geometrical optics, but still could not find out whether the four arrangements in question had already been discovered. The truth is, the knowledge he was after was well known in the late 19th century, but it vanished in our age, except in such non-trivial sources as Coxeter's chapter on polyhedra in Rouse Ball's book, Shubnikov and Koptsik's monograph on symmetry, and Courant and Robbins's introduction to mathematics. We recommend that interested readers have a look at Walker's paper because of his impressive illustrations and clear explanations.

Walker's paper reflects, however, an entirely new phenomenon in connection with the kaleidoscopes, namely the interest of contemporary artists and inventors in the topic. There are several new versions of the classical kaleidoscope, and even interesting marriages with modern technology, using systems of lenses, flashing diodes, and other devices. Of course the kaleidoscope is not a universal tool for high-tech art, but it has gained some application. For example, the *karascope*, which has a curved reflecting sheet inside the tube and uses a polarizing filter, was patented not long ago by Judith Karelitz. It was commissioned by the New York Museum of Modern Art, where the device is on sale. Even Brewster's idea of using the kaleidoscope for color-musical performance has got a new impact with a special *projection kaleidoscope*, or kaleidograph, designed by Don Anderson (1975), the director of the industrial laboratory of the Eastman Kodak Company. In this system the classical device is connected to a loudspeaker in such a way that the sound waves are able to jiggle the light colored fragments in the object box of the kaleidoscope. Furthermore Anderson upgrades the system by polarizing filters and other tricky ideas. Finally, using a projection lense, the patterns will "dance" synchronously with the music, changing both form and color. May we add that the most light organs, used for example by the Russian composer Skryabin in early 20th century, focus just on the color association of the music. There was, however, a movement pioneered by Alexander Laszlo in the 1920s, during his performance at the Festival of Kiel, Germany, which aimed to find the harmony of form, color, and music (see his monograph *Die Farblichtmusik*, Leipzig, 1925). Although the entertainment industry, from disco bars to rock concerts, rediscovered the colored music, but the theoretical and aesthetical studies are missing (the interdisciplinary works of the scholars and artists around the Prometei [Prometheus] Studio led by Bulat Galeev in Kazan', U.S.S.R., are the rare exceptions). Returning to kaleidoscopes, the idea of using various lenses and polarization effects also appeared in commercially marketed kaleidoscopes. Visiting some shops we saw, for example, the *bubblescope* the *polarizescope*, and other Hong Kong and Taiwan made devices. Interestingly, most of these technical kaleidoscopes do not go far beyond Brewster's classical arrangement of mirrors, but still provide many new phenomena. Obviously the kaleidoscopes of Möbius, Fedorov, Hess, Coxeter, Shubnikov and Koptsik would give further possibilities in a marriage with technology. The "ars mathematica" of Nikolaus and Caspar Schwabe (1986) represents an interesting chapter in the history of the kaleidoscope, because they also deal with trihedral kaleidoscopes (cf., Fig. 4 of this paper). They constructed various large-scale models and walk-in-kaleidoscopes (Darmstadt Symmetry Exhibition, 1986; Farnborough, 1988; Frankfurt, 1989). The Gallery and Shop AHA in Zürich, Switzerland, co-founded by the Schwabes, also deals in kaleidoscopes. Another interesting project was worked out by Vedder Wright (1989), an American specialist in spatial design, who gave classes for elementary school students about the symmetry of Platonic solids using trihedral kaleidoscopes. In Japan the artist-inventor Naoki Yoshimoto, the discoverer of the Yoshimoto cube, also deals with kaleidoscopes. The designer Naomi Asakura (1990) makes experiments with both plane and curved mirrors. It is interesting to note that the American inventor Buckminster Fuller (1979) also deals with the same foldable layout which is associated with the Coxeter kaleidoscope (Fig. 4), but he uses it for a different purpose as one of his quanta modules. In other cases he refers to kaleidoscopes metaphorically or, at least, semi-metaphorically.

Another analogy between the old and the recent period can be seen in the book market. Brewster wrote an entire monograph on the kaleidoscope soon after its invention. Moreover, in old age, almost 40 years later, he returned to the same topic, revising and extending the earlier work under a new title (Brewster, 1819; 1858). Although this second book had a further edition in 1870, later the kaleidoscope, as an optical device, disappeared from the titles of books (but not in a metaphoric sense in the title of very many collections, anthologies, and some periodicals). More recently, however, the kaleidoscope again became a topic for books. We are aware of two recent monographic surveys of the optical device, although we do not have access to them (Baker, 1985; Yoder, 1988). Another book discusses how to create kaleidoscopic designs (Finkel and Finkel, 1980). There are also other items available on the topic, including a thesis on the use of kaleidoscopic patterns in elementary art education (Costello, 1988), another one using the kaleidoscope for discussing culture-theoretic questions (Gray, 1989), as well as a great number of popular articles for children and amateurs. Even the expression "kaleidoscope" inspired further technical terms in both Brewster's time and more recently. Let us start with the early period. Charles Wheatstone, who was 15-year old when Brewster patented his kaleidoscope in 1817, invented a decade later an instrument which he called *kaleidophone*, or *phonic kaleidoscope* (Wheatstone, 1827). The young Charles was also inspired by his family's profession of making musical instruments which directed his interest to acoustical experiments. The kaleidophone is a device for visual display of sound-waves: it is based on the illumination of the free end of a vibrating rod. Wheatstone became also interested in Chladni's sand pattern technique which produces symmetric patterns on solid plates vibrated, for example, by a violin bow at their edges. This approach gained some prominence in more recent experiments (cf., Mary Walker's book *Chladni Figures: A Study in Symmetry*, London, 1961). Wheatstone's name is, however, better known for his later studies in electronics, in particular in connection with the Wheatstone bridge for measuring resistances (originally invented by Samuel Christie). The second flowering of kaleidoscope-related terms is associated with the more recent period. On the border of art and mathematics such expressions were coined as *kaleidocycles*, that is kinetic polyhedra made from chains of tetrahedra (Schattschneider and Walker, 1977), and *kaleidometrics*, which is a method of drawing kaleidoscopic patterns with special grided circles (Shaw, 1982). Interestingly, the term *kaleidophone* was also reintroduced in a new sense in 1984 by the Prometei Studio in Kazan'. Their kaleidophone is an optical device for enriching musical performances by light effects. After seeing so many modern artistic applications, we can no longer reject so strongly Brewster's dream and the related myth about the artistic usefulness of the kaleidoscope. Sometimes a myth is a herald of a later chapter...

Kaleidoscopes, mostly the well-known classical ones, are also mentioned in some interdisciplinary works on symmetry. Here we refer only to the recent monograph by Werner Hahn (1989) which has an especially broad spectrum with an unbelievably large number of illustrations. During his study of symmetrization in space Hahn uses dihedral kaleidoscopes to multiply some basic shapes. This chapter, which also emphasizes the importance of light during the experiments, includes photographs. In a later chapter the artist-biologist author characterizes his *ars evolutoria* as "kaleidoscope-related genesis-processes". Hahn remarks that in this sense his works give a counter-example to the negative views expressed by Gombrich about the exploration of kaleidoscopic shapes. Probably Gombrich is right from the geometrical point of view: these shapes, as we discussed in Part 1 of this paper, are so simple that we would have little joy in exploring their structures. However Hahn's approach of looking for an analogy to evolutionary transformations of organic shapes is more biological. A classical chapter of embryology focused exactly on the splitting of fertilized eggs and their connections with the later symmetry of the organisms. This topic also aroused the interest of the mathematician Hermann Weyl who discusses related questions in his book *Symmetry* (Princeton, 1952, pp. 33-38). A new impact came in the mid-1970s when Richard Gordon and Antone Jacobson suggested a computer simulation technique, based on the distortion of a geometric grid, which "performs" experiments that cannot be done in biological laboratories (see their article in *The Journal of Experimental Zoology*, 197 [1976], 191-246 and the survey in *Scientific American*, 238 [1978], No. 6, 106-113). Some of the simulated images even have aesthetical value. Returning to Hahn's monograph, he also refers to computer graphics as a field where kaleidoscopic images may have an importance. Moreover, he is not the only one who combined artistic and biological interests in connection with the kaleidoscope: Thomas H. Huxley, as we have already mentioned in Part 1, also referred to the optical device.

THE PERIODIC TABLE OF KALEIDOSCOPES AND KALEIDOSCOPIC GROUPS (KALEIDOSCOPIANA SYMMETRICA)

During our kaleidoscopic excursion around the kaleidoscope we have seen many kinds of instruments. It is probably useful to summarize them, moreover, to present a classification that would be in "harmony" with the presented symmetries (symmetry groups). This list - or, with an analogy to chemistry, periodic table - considers all the possible devices formed with plane mirrors and the related groups, which we may call kaleidoscopic groups (the conventional name is reflection groups). In the right column the terms variable versus stable, that is the possibility versus impossibility of a hinged edge, refer only to the types of kaleidoscopes, while the number indicates both the number of types of kaleidoscopes and the number of symmetry groups (see more details later):

Kaleidoscopes according to the arrangement of mirrors (number of mirrors)	**Presented patterns (symmetry groups)**	**Number of types of kaleidoscopes (symmetry groups)**
— dihedral ($m = 2$)	rosettes (planar point groups)	infinitely many variable
special case (two parallel)	frieze (frieze group)	one stable
— cylindrical ($m = 3, 4$)	wallpaper (plane groups)	four stable
— trihedral ($m = 3$)	spatial rosettes/polyhedra (spatial point groups)	infinitely many variable and three stable
special case of variable (U-shaped)	frieze (frieze group)	one stable
— trough ($m = 4$)	strung pearls (rod groups)	infinitely many variable
— box without front and top ($m = 4$)	two-sided frieze (two-sided frieze group)	one stable
— open box ($m = 4, 5$)	two-sided wallpaper (two-sided plane group)	four stable
— polyhedral ($m = 4, 5, 6$)	spatial periodicity (space groups)	seven stable
— variable trihedral, box without front and top, and open box — with colored bottom	rosettes, friezes, and wallpapers doubled in another color (gray groups)	[as above in the corresponding cases]
— determined by appropriate colored symmetric object	color change periodically (colored symmetry)	[usually just one stable arrangement]

In the case of determining the types of kaleidoscopes (right column) of a given kind (horizontal row), we do not distinguish between those arrangements of mirrors where the corresponding angles are the same. In other words, two kaleidoscopes are considered to be of the same type if their arrangements of mirrors are equivalent by an appropriate angle-preserving transformation. Such a transformation is called conformal; here we also require that straight lines should not be curved (i.e., the transformation is also affine), because all the kaleidoscopes discussed here are formed by plane mirrors with straight edges. Although this classification may seem too sophisticated, it is really a natural convention, which is used directly or intuitively in many fields. For example, in the terminology of elementary geometry, moreover in colloquial language, such words as rectangle and rectangular right prism ("brick") refer to infinitely many figures with various sizes and proportions; only the corresponding angles remain invariant. One of the earliest laws of crystallography, which was discovered by Niels Stensen (Nicolaus Steno) in 1669, states the constancy of angles for the crystals of the same material. The biological growth of individual bodies often can be characterized by conformal transformations, where the angles are invariant, but the lines can be curved. The fact that the proportion of the human body changes significantly from childhood to adulthood — that is the transformation is more general than just a mere similarity (enlargement) — has been investigated by many artists since the Renaissance. Albrecht Dürer worked out an entire set of canons following this process. D'Arcy Thompson in the 1910s generalized and systematized this idea in theoretical biology, also discussing botanical and zoological examples. This work inspired, among others, the aforementioned simulation model for embryology by Gordon and Jacobson. The pursuit curves in the three-bug problem, which were discussed in Part 1 in connection with the image on the cover of this volume, are derived by an angle-preserving process. Sergei Petukhov investigated conformal symmetries — properties which are invariant under conformal transformations — in the case of growth and motion of biological objects, and he applied the findings in mechanical engineering. More general conformal symmetries also play significant roles in particle physics.

Returning to our kaleidoscopes, we can clearly see two sorts of types. In the cases of all the dihedral kaleidoscopes, a class of trihedral ones, and all the trough-shaped ones, the types differ only in one angle ($180°/n$, where n is a variable). In these cases the arrangement of mirrors has some variability, so a specific type can be easily transformed into other ones just by rotating a mirror around an edge (cf., the idea of the polyangular kaleidoscope and our earlier note in the caption of Fig. 1 in Part 1). Other types of kaleidoscopes are stable in some sense: changing one angle is not enough to form another type. Note that in case of the partial kaleidoscope (Fig. 3), which is a simplified version of the stable trihedral kaleidoscopes where a part of the surface is missing, there is variability; this is precisely the advantage of the device over the complete trihedral versions. Obviously, the property of variability or stability only refers to the types of kaleidoscopes, while in the case of symmetry groups, or repetition types of patterns, just the numbers are important. The table gives in each row the specific number of kaleidoscopic groups, that is, of those symmetry groups which can be generated just by reflections. Among the 17 wallpaper groups and 230 space groups there are four and seven kaleidoscopic groups, respectively; these can be fully represented by kaleidoscopes. In many other cases, where at least one reflection is involved, we may represent the local arrangements (point-groups) of ornamental or crystal structures.

An important advantage of this classification of kaleidoscopes is its correspondence with the group theoretic characterization of the presented patterns. It is manifested by the double function of the right column. Specifically, all of the kaleidoscopes which belong to a given type present patterns of the same symmetry group (allowing any kind of asymmetric object), while different types represent different groups. This further supports the argument that - contrary to Shubnikov and Koptsik - we do not consider the trihedral kaleidoscopes of (90°, 90°, 180°/n) as one type, but as infinitely many variable ones (cf., Fig. 1 in Part 1). Of course this theoretical number is severely limited by the imperfections of the mirrors. We should also remark that not all of the obtained types of rosettes (point groups) can occur in ideal crystal-structures: the periodicity (lattice-likeness) allows only 1-, 2-, 3-, 4-, and 6-fold rotational symmetries. Real crystals, however, may depart from this strict rule; see the recent developments in connection with the 5-fold symmetric quasicrystal. Some dihedral, trihedral, and trough-shaped kaleidoscopes can present 5-fold symmetric finite patterns. Now these structures can be associated not only with the organic world, but also with special crystal-structures.

Of course we should again emphasize, self-critically, that the simple descriptions cannot replace the real joy of looking into these kaleidoscopes. If a picture is worth a thousand words, as the ancient Chinese proverb claims, the kaleidoscope presents a multiplication of a thousand words: almost a whole booklet on geometric crystallography and related subjects. To avoid further frustration, it should be remarked that we plan a more detailed survey of these kaleidoscopes, as well as the organization of a workshop on the topic with an exhibition mounted during the forthcoming *Interdisciplinary Symmetry Symposium and Exhibition* (1992); see also the "Call for kaleidoscopes" in this quarterly (Vol. 1, No. 1, p. 93).

We have visited many fields, from optics to design, from geometry to crystallography, emphasizing the educational aspects, during this rather kaleidoscopic and incomplete survey of the kaleidoscope. Moreover, we have also touched on linguistic, poetic, aesthetic, psychological, as well as practical questions, and also covered some relevant events in the history of science. At this point we should pay tribute to the early pioneer of the topic, Sir David Brewster (1781-1868). He was a very productive man, not only as a kaleidoscope-maker and popularizer of science, but also as a researcher, author of monographs, editor of encyclopaedias and scientific journals, and finally as a father (he was in his 80th year when his younger daughter was born of his second marriage). His major scholarly interest was in optics (cf., the terms Brewster angle of polarization, Brewster fringes), but he also contributed to the theories of magnetism and of hydrodynamics. He has a strong record of works on the history of science, including an often quoted biography of Newton, another one on the three pioneers of modern astronomy Galileo, Brahe, and Kepler, as well as some publications on the history of optics. Other works by him deal with religious questions, the history of Freemasonry, and numerous other topics. Brewster kept in close contact with many eminent scholars and artists of his period, including the astronomer Herschel, the pioneer of photography Talbot, and the poet Sir Walter Scott; he visited in Paris, among others, Arago, Berthollet, Biot, Laplace, and Poisson. Brewster's father-in-law was the poet "Ossian" Macpherson. While the real authorship of the "ancient" Ossian-poems remained secret for a long time, Brewster solved a similar problem in science: he proved that the "discovery" of the correspondence between Newton and Pascal was a simple case of forgery.

From the point of view of our subject it is also interesting to mention Brewster's studies of the forms of crystals and of the figures of equilibrium in liquid films, which are strongly connected with the later interdisciplinary studies of symmetry. One of the earliest comprehensive monographs on the subject, F.M. Jaeger's *Lectures on the Principle of Symmetry* (Amsterdam, 1917), refers to him in connection with the optical anomalies in crystals, but surprisingly misses the subject of the kaleidoscope. Brewster contributed significantly to the modern approach of symmetry and to the spread of the term "symmetry". We should learn from the interdisciplinary "excursions" of Sir David and many other pioneers in this survey. In their cases the interdisciplinarity was not only a slogan, but real practice. Unfortunately this component is often missing in our age, especially in the broader spectrum of arts and sciences.

The author is indebted to György Darvas, John Hosack, and Jarek Woloszyn for various discussions about this paper. In connection with references see the bibliographic article "Kaleidoscopes in art, science, and education" in the section *Symmetro-graphy*.