SYMMETRY SETS OF PLANE CURVES

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Let $\alpha : S^1 \longrightarrow E^2$ denote a piecewise C^3-regular curve in the plane which may have a finite number of transversal selfintersections. The *symmetry set* S_α of α is given by the closure of the set

$$\tilde{S}_\alpha := \{p \in E^2 \mid L_p \circ \alpha \text{ attains the same critical value at least two times along } \alpha\},$$

where L_p denotes the distance function with center p. Through the corresponding points p_1, p_2 on α there passes a *bitangent circle* of α with center at $p \in \tilde{S}_\alpha$; p is called a *center of local symmetry* of α. There is an axis l of *reflectional symmetry* through p such that p_1 and p_2 as well as the tangents to α at these points are symmetric with respect to l. The collection of these axes is called the *dual* of the symmetry set.

In the case of a closed curve α the subset Z_α of S_α given by the closure of the set

$$\tilde{Z}_\alpha := \{p \in E^2 \mid L_p \circ \alpha \text{ has at least two absolute minima}\},$$

is of particular interest. It is called the *medial axis, skeleton* or *cyclographic set* Z_α of α. It mainly consists of bisupporting circles of α. These circles may be used to reconstruct α from Z_α as an envelope.

A report is given on the local structure of S_α in the generic case. Furthermore the development of this set and its dual during a deformation of α by a 1-parameter family of curves is described. This is is used to establish a global theorem concerning the number of characteristic data of symmetry sets. All these results are valid in the generic case only. They can be transfered to the piecewise linear case by establishing suitable interpretations of the data defined in the smooth case. Also the case of smooth piecewise circular curves deserves an extra mention. References for this kind of research can be found in the work of Gibson, Giblin, Bruce, Tari and Banchoff.

Ends of the medial axis correspond to vertices of plane curves. It is explained how this can be used to prove Bose's vertex theorem in the case of simply closed smooth curves. (The most general version of Bose's theorem can be found in the work of Haupt in terms of order geometry.) Interpretations of curvature different from those due to Banchoff and Giblin lead to a version of Bose's vertex theorem in the piecewise linear case. These interpretations deal with osculating circles and extremal values of their radii. They are motivated by the study of parallel polygons (offsets). The main tool for the proof again is given by the medial axis.

Finally, methods for the computation of the medial axis are dicussed. They developed from the computation of the Voronoi cells of finite point sets. The most powerful method for closed polygons is due to S. Meyer. It is based on a geometric extension of the sweepline method given by Fortune for Voronoi cells. Its complexity is of order $O(n \cdot \log(n))$. In particular, it provides a fast approximation of symmetry sets in the smooth case. This is useful for the many applications of symmetry sets. They have been used by Blum and Thompson for the description of shape. Furthermore, they provide a good tool for the storage of graphic data which should be rescaled or manipulated in some other way. In particular, they are the basis for the description of families offsets and boundaries of growth processes. Also a method is mentioned how to use symmetry sets for the smoothing of graphic data.

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