TRISECTING THE CUBE: NEW OBSERVATIONS

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In the early 1970s, my design students were introduced to the task of sectioning the cube into any possible number of congruent, superposable pieces (2, 3, 4, 6, 8, 12, 16, 24). Subsequent to a slip in Martin Gardner's monthly *Scientific American* column (September 1980), the *trisectioning* of the cube became a focus of one ongoing study. For two decades, one strategy for thirding the cube, characterized as a *puckered* or *trussed* class, was exhaustively examined. The inventiveness of the students produced a rich array of solutions. Among the great diversity of *puckered* trisections, one trisectioned cube that, in fact, presented a different strategy—one that characterizes a *prismatic* or *extruded* class—went unidentified for many years. In due course, more strategies were identified, bringing the number of classes to six: third, *radiant* or *pyramidal*; fourth, *snub-pyramidal*; fifth, *helical* or *twist-truded*; sixth, *equatorially disjointed*. In the fall of 1997, Xin-hua Jian, a graduate student, made a significant observation: The *equatorially disjointed* class was only a special case; it belonged to a larger *longitudinally layered* class. He showed, as well, how this more inclusive class relates to the *twist-truded* class.

Excepting the cube that is simply sliced into three flat slabs (envision butter pats!), all trisections are essentially deformations of a basic trisection [Fig. 1] known, according to Gardner, in ancient China—where the triplets were called "yangmas." Focused on the singular Chinese curiosity, Gardner made a too casual remark: "I know of no third way to cut a cube into three identical parts."

A trisectioning strategy involves two major considerations: (1) how the three continuous traceries (emanating from one vertex of a cube, traversing in threefold synchronization the cube's edges or faces, and converging on its antipodal vertex) are contoured; (2) how the interior planes of trisected pieces are resolved. What happens on the interior is far from evident from exterior traceries. Strategies that varied substantially in either of the two factors led to our designation of the distinctive classes. Abbreviated descriptions of six classes follow. Note: The date of a student's example does not necessarily date the identification of its class.

Puckered or *trussed*: More or less any contour can be drawn across both the upper and lower faces of the cube. (*Upper* and *lower* pertain to the vertical orientation of the cube in respect to one of its diagonals [one of four threefold axes], the anchor of the means of trisectioning; in all classes, this operative axis remains intact, after trisectioning—an unbroken line that all three pieces share in common.) Located on the operative axis, a single *point* or multiple *points of resolution* expedite a truss-like triangulation of interior surfaces with each straight line segment of a tracery (curved segments produce conical surfaces). At times, two triangles can lie in the same plane, forming square or other quadrangular faces [Fig. 2, Marlowe, 1978]. The maximum number of *points of resolution* are two less than the number of line segments of a tracery [Figs. 3, one point of resolution, shifted off center; 4, two points; 5, three points, with one at center, Polletta, 1983; 6, four points]. If traceries are too convoluted, the trisected pieces, unless pliable, cannot be assembled or disassembled. An additional strategy of deformation for this and other classes can

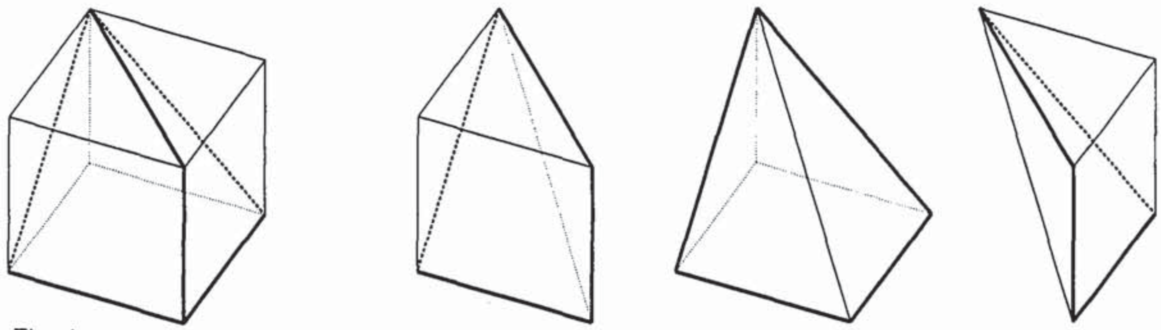

Fig. 1

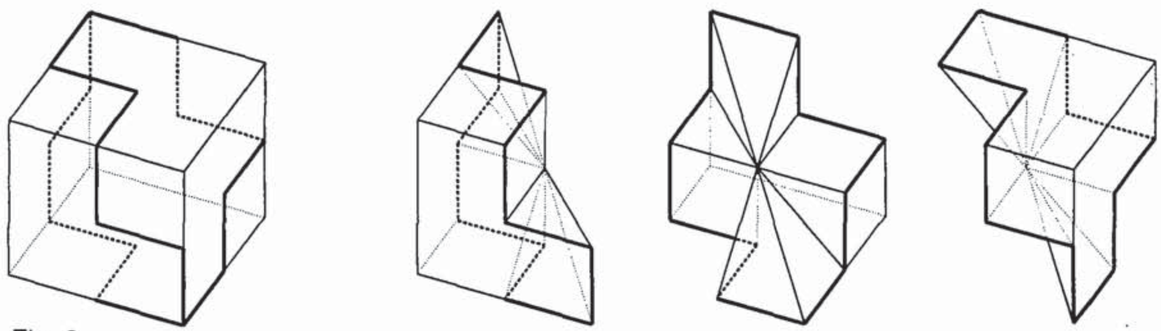

Fig. 2

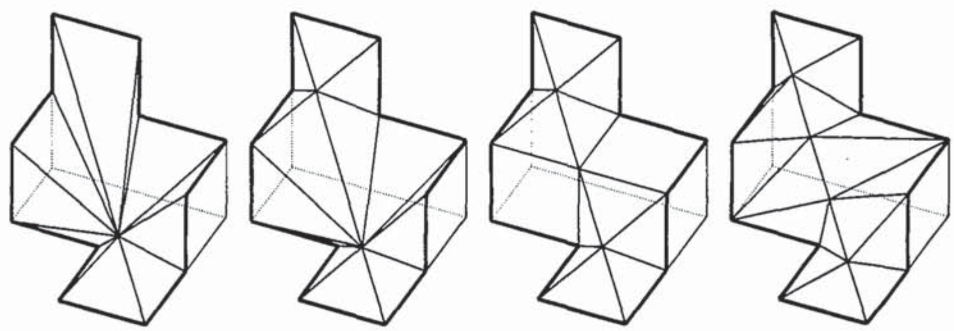

Fig. 3, 4, 5, 6

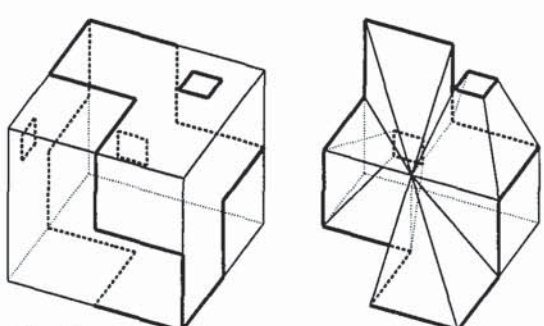

Fig. 7

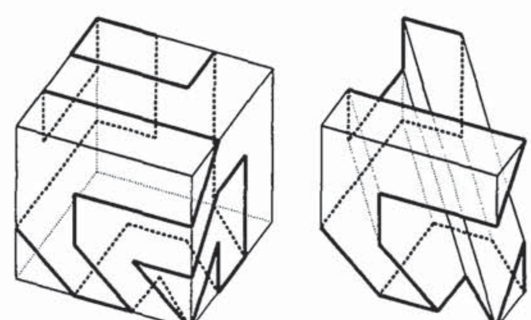

Fig. 8

be characterized as *excavate and mount*: Extract a part of the mass of a trisected piece from one face and adhere it to its reciprocal face. This give-and-take can bring about penetrations of the cube's exterior faces [Fig. 7, Gibbons, 1993].

Prismatic or *extruded*: From each line segment of the traceries that are delineated on the three faces around one polar vertex of the cube, planes, all parallel to the operative axis, pass through the cube and intersect the three opposing faces [Fig. 8, anonymous, 1990]. Only the traceries on either the top or bottom set of faces can be designed; those on the antipodal faces are projectively determined by their counterparts. Traceries of this class can be extremely convoluted, including intricate curves; yet the pieces can always be slid apart along the operative axis.

Radiant or *pyramidal*: Planes, all triangular, radiate from one point; that point is one of the two vertices of the cube through which the operative axis passes [Fig. 9, Kupinski, 1987]. The three traceries, branching from the radiant vertex, must travel either straight across the abutting faces or continuously along the abutting edges until they meet the boundaries that are contiguous with the antipodal faces; from there, the traceries can meander (in threefold sync) in comparative unrestraint.

Snub-pyramidal: Triangular planes, radiating from one point on the operative threefold axis beyond the surface of the cube, are resolved in a similar manner to the former class. However, the traceries on the near faces of the cube derive some complexity from the projective intersections of the radiating triangles with the those faces [Fig. 10, alteration of Kupinski's *pyramidal* trisection]. The configuration of those traceries correspond, of course, to the relatively unrestrained design of the traceries on the faces of the far side. All interior faces are quadrangular, excepting the two triangles that sit at the two outer flanks of the pleat of quadrangular faces.

Helical or *twist-truded*: In reflecting on the double-helical staircase at Chambord (presumed Leonardo's design), there seemed to be no geometric impediment to a triple-helical sectioning of the cube [see Fig. 14]. If the rate of helical twist is substantial, the only way to assemble and dissemble the cube is to screw the pieces into and out of place. The traceries are governed by the rate of twist.

Equatorially disjointed: If a cube is sliced into two congruent parts through one of its four hexagonal (equatorial) planes—each normal to one of the threefold axes, both of the halves retain threefold symmetry and can be sectioned independently of one another (thus differently) into three congruent parts. If, then, a trisected part of one of the halves is rejoined to a trisected part of the other half at an overlap of the surfaces that they commonly share on the equatorial plane, the resultant single piece will be a proper trisection of the cube [Fig. 11, Desta, 1993].

Longitudinally layered: In recognizing that the *equatorially disjointed* trisection is a special case, Jian showed that longitudinal slices, parallel to an equatorial plane, can be made anywhere—and not of necessity at the equatorial plane. Each layer, retaining threefold symmetry, can be independently trisected with respect to the other layers; a set, then, of trisected pieces, one from each layer, can be sequentially rejoined to one another to make up a trisection of the cube [Fig. 13, five layers, no slice at equatorial plane, Jian, 1997]. With Jian's identification of the *layered* class, one trisection from prior years was identified as an example of it [Fig. 12, Baccari, 1978]. Jian also saw that if each thirded slice was of equal thickness and rotated by a regular increment, the resulting assemblage would be *helical*. Though the illustrated example [Fig. 14, Jian, 1997] is segmented into 40 layers, the stepped inner surfaces must be smooth, if the three pieces are to be snugly screwed into, or apart from, their intended cubic configuration.

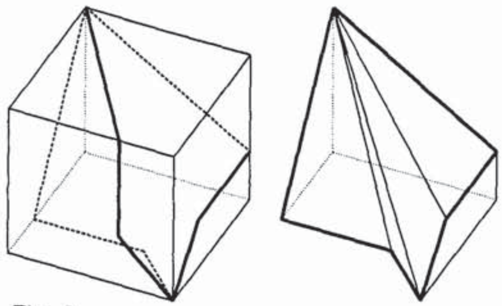

Fig. 9

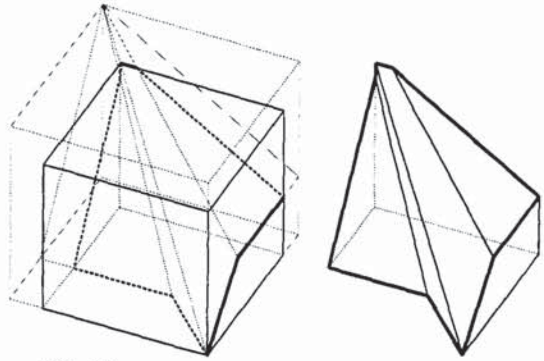

Fig. 10

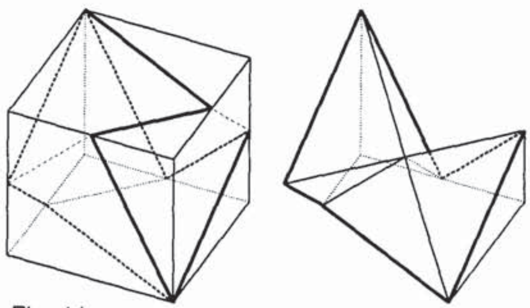

Fig. 11

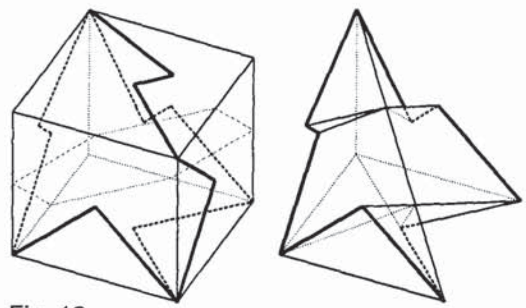

Fig. 12

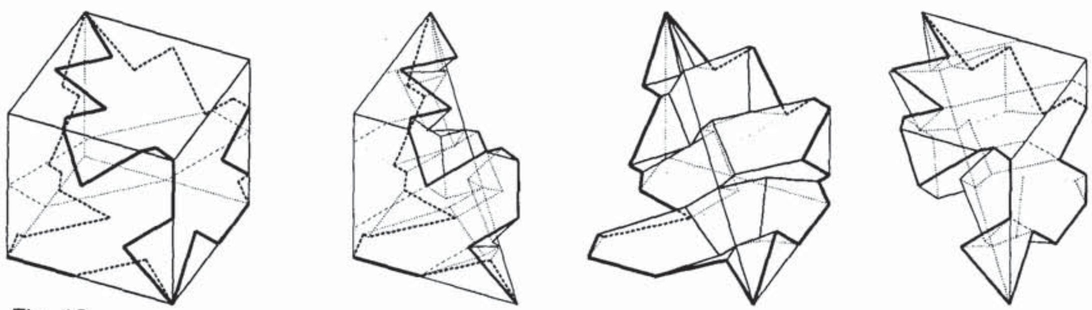

Fig. 13

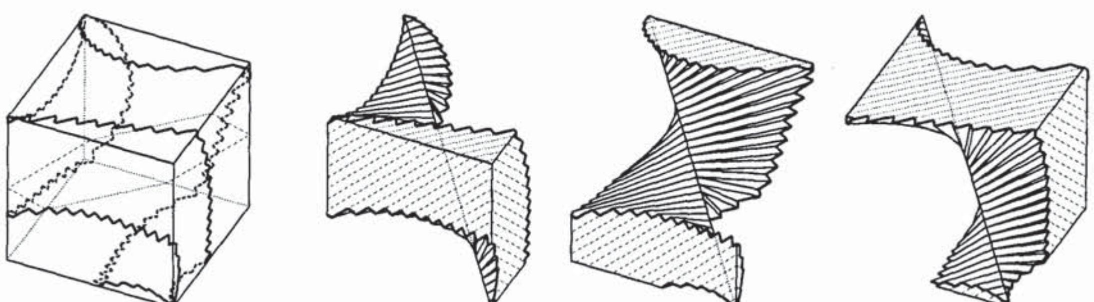

Fig. 14