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The possibility to study knots from the mathematical point of view was for the first time proposed by C.F.Gauss. Gauss formulated the "crossing problem", by assigning letters to the crossing points of a self-intersecting curve and trying to determine "words" defining a closed curve. J.B.Listing represented knots by their projections (diagrams) and made an attempt to derive and classify all the projections having fewer than seven crossing points. The complete derivation of non-isomorphic knot projections having fewer than ten crossings was completed by P.G.Tait and T.P.Kirkman [1]. Kirkman's geometrical system for the systematic derivation of knot projections, closely connected with the enumeration of polyhedra, represented at the same time the geometrical method for the classification of knot projections.

In the 30-ties, after the appearance of the first knot invariant, discovered by J.W.Alexander, the knot theory was established as the part of topology, completely loosing connection with it's roots - geometry. In K.Redmeister's book "Knotentheorie" (1932), each knot is represented by one projection, (randomly?) chosen from several possible ones. Today, with the development of computers, the enumeration of knots and links is very similar with the situation occurring in different unordered structures: prime numbers, polyominoes etc., giving no chance for any classification. Following the "geometrical" line (Kirkman-Conway-Caudron), we will try to present the consistent (geometrical or graph-theoretical) approach to the derivation and classification of knots and links.

In this paper will be considered only prime alternating knots and links, denoted by Conway notation [2]. For the both of them we will use the general term "link", considering knots as 1-component links. A prime link with singular digons, expressed by a Conway symbol, is called generating. Any other link could be derived from some generating link, by replacing singular digons by chains of digons. All links that could be derived from a generating link by such replacement make a family. All links are distributed into disjoint sets, called by A.Caudron worlds [3].

The most powerful tool in our consideration is the symmetry. It could be visualized representing links by graphs, and trying to imagine them as 3D illustrations - maps on a sphere. Every 4-regular planar edge-connected graph uniquely defines an alternating link. If digonal edges are denoted by colored (bold) lines, then among bicolored graphs (including uncolored ones) representing generating links, we distinguish regular and combined graphs: 3-regular (where in each 3-valent vertex there is exactly one colored edge), 4-regular, and combined 3- and 4-valent graphs. A generating link is called basic if its bicolored graph is regular. If it is 4-regular, such graph is a basic polyhedron. The term "graph of link" means "bicolored graph of link".

For every alternating link we could distinguish the symmetry group G of its bicolored graph, and its subgroup G' of index 2 (or antisymmetry subgroup G/G'), obtained by alternating, representing the actual symmetry of a link.

The origin (or O-world) is the basic polyhedron 1 – the 4-regular graph with one vertex, a usual symbol of infinity. The first, linear world (or L-world) contains only one basic link: a digon 2. From it, we derive the infinite family p, consisting of p-gons with digonal edges (p>2). For p even we have the infinite series of knots 3, 5, 7,& (or 31, 51, 71,& ), and for p odd the infinite series of 2-component links 2, 4, 6,& (or 212, 412, 612,& ). For each of them, the Alexander polynomial Dδ for knots, or reduced Alexander polynomial Dδ(t,t) for 2-component links, is an alternating polynomial of order p+1, with all coefficients equal to 1. For every knot p of this family, the unknotting number is u(p)=(p-1)/2. All knots of L-world are periodical knots with graph symmetry group G=[2,p], and with knot symmetry group G'=[2,p]+, generated by p-rotation and 2-rotation. The periods of every such knot are p and 2.

The second, rational world (or R-world) consists of rational links. Its basic links are 2l (l>1) (Fig. 1). The number of rational knots and links is 2n-4+2[n/2]-2. Using the combinatorial methods, it is also obtained the recursion formula for the number of generating rational links. Because generating links are completely sufficient, we could restrict our attention to them, and to infinite families generated by them. Let us consider the first nontrivial infinite family of rational links, generated by 22 (or the knot 41). Its graph is the tetrahedron with two colored nonadjacent edges. From it, we obtain the infinite family pq, consisting of [n/2]-1 links for every n fixed (n=p+q).

Figure 1.

The joint properties of links belonging to this family and their symmetrical distribution are illustrated by Fig. 2. In this table, every knot and 2-component link is given also in standard Alexander&Briggs notation, consequently extended to the knots with more then 10, and 2-component links with more then 9 crossings. For every knot is given its Alexander polynomial Dδ(t), and for every 2-component link its reduced Alexander polynomial Dδ(t,t), both abbreviated thanks to their symmetry. In the case of knots, a0+a1+& +ak means a0+a1t+& +aktk+& +a2kt2k, and for 2-component links a0+a1+& +ak means a0+a1t+& +a2kt2k+a2k+1t2k+1. For every knot, in the corresponding upper left corner is given its unknotting number, and amphicheiral invertible knots are denoted by "f" in upper right corner. In the family pq we have knots, and 2-component links for p=q=1 (mod 2). The period of the knot 22 is 2, and the same holds for all of the family pq.

Figure 2.

After R-world, we consider so-called "prismatic" or stellar world (S-world). There we could distinguish the basic links 2,2…,2 (S-links) and the direct products of basic links 2,2,…,2+ (SxO-links), 2,2,…,2++ (SxL-links). From them originate the corresponding subworlds of S-world, all generating links belonging to them, all different links and their infinite families, each of them defined by a generating link.

For every even n (n>5) we have (n/2)-component basic link 2,2,…,2. Its graph is an n-gonal prism with colored lateral edges. Its graph symmetry group G=[2,n] is generated by n-rotation, vertical and horizontal plane reflection. From this, we conclude that 2,2,…,2 remains invariant after cyclic permutations of digons (rotations), where every such permutation is identified with its obverse (because of vertical reflection), or if all digons are obverted (horizontal reflection).

From a basic S-link 2,2,…,2, we derive generating links replacing digons by rational tangles not beginning by 1. Continuing in the same way, by such "rational" replacements we derive all generating knots and links belonging to "stellar-rational" world and their infinite families. The next world is arborescent (or A-world). Its members are the multiple combinations of links belonging to the preceding worlds, etc.

Probably the most interesting world is the polyhedral or P-world. Here, the first problem is the derivation of the polyhedra: 4-regular graphs without digons. This problem is solved by Kirkman [4] for n<13, where the polyhedra are obtained by introducing a new triangular face in the link diagrams, in order to eliminate all digons. To the polyhedrons we could apply Polya Enumeration Theorem (PET), in order to calculate the number of source links derived from each polyhedron by introducing digons in it's vertices. After this, we could continue with the series of the replacements, in the same way as before, and obtain the links of polyhedral-rational world, etc.

Using geometric-combinatorial methods and the classification proposed, the complete list of prime alternating links with n<13 crossings, is derived. All the links obtained, belonging to the corresponding subworlds and given in Conway notation, are classified in infinite series of links - the families. Combinatorial formula giving the number of rational links for every n is derived, as well as some combinatorial results for the number of polyhedral source links for n<13. Amphicheirality is recognized from the antisymmetry of vertex-bicolored graphs. The complete list of knot and link projections, divided into the corresponding families, is obtained for n<10.

References:

[1] Kirkman, T.P.: The enumeration, description and construction of knots of fewer than ten crossings, Trans. Roy. Soc. Edinburgh, 32 (1885), 281—309.

[2] Conway, J.: An enumeration of knots and links and some of their related properties, in Computational Problems in Abstract Algebra, Proc. Conf. Oxford 1967 (ed. J.Leech), Pergamon Press, New York, 1970, pp. 329-358.

[3] Caudron, A.: Classification des noeuds et des enlancements, Univ. Paris Sud, Dept. Math., Public. Math. d'Orsay, 82, Orsay, 1982.