Symmetry and Insight: The Saga of Sphere Eversions

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We discuss the role of symmetry and of modern computer visualization in the understanding of a problem that has stimulated the imagination of mathematicians and non–mathematicians alike: how a sphere can be turned inside out without tears or creases. The contribution of both symmetry and computer visualization is made clear in the context of the computer animation *Outside In*, directed by the authors and Tamara Munzner. This talk is meant for general audiences. In the *Art and Visual Mathematics* workshop Silvio Levy will present more on the mathematics of the sphere eversion.

A Fascinating Puzzle

In 1957, the young mathematician Steve Smale made a startling discovery: you can turn the surface of a sphere inside out without making a hole, if you think of the surface as being made of an elastic material that can pass through itself. Communicating how this process of *eversion* can be carried out has been a challenge ever since.

Smale's proof that this could be done relied on a large number of small geometric steps that, although "constructive" in the mathematical sense, are not much help if one's goal is, for example, to make an animation of the sphere eversion. Smale's proof is somewhat like a research paper describing, in minute detail and down to the molecular chemistry, what happens to the ingredients of a souffle during the cooking process: one cannot expect a person who has never seen a souffle to follow this "recipe" in preparing the dish.

In 1961, Arnold Shapiro invented the first explicit eversion, but he did not publish or divulge it widely. The first time that most mathematicians and the public at large became aware of an explicit eversion was when Tony Phillips worked out the details of what he thought was Shapiro's eversion (though later it turned out to be a different one). Phillips published a beautifully written article in *Scientific American*, aimed at a wide audience and culminating with a series of pictures representing various stages of the evolution. The publication of these pictures dispelled the mysterious and paradoxical reputation of the problem; but filling in the missing levels in Phillips' pictures and tracking them through their implied deformations was not an easy task. People began searching for simpler solutions.

The French mathematician Bernard Morin devised in 1967 a new eversion that was simpler than preceding ones, in part because the sphere preserved a certain amount of symmetry throughout the process. Morin, incidentally, is blind, and the fact that he was one of the first people to understand how a sphere can turn inside out is both a tribute to his ability and a convincing proof that "visualization" goes far beyond the physical sense of sight.

Charles Pugh made wire–mesh models of various stages of Morin's eversion, and Nelson Max digitized these models and used them as the basis for the movie *Turning a Sphere Inside Out*, an early triumph of computer animation. The rendering of the evolving sphere was done by Jim Blinn.

Other methods were found later. In the mid–seventies, Bill Thurston developed his idea of corrugations, a technique that allows "mathematical materials" like curves and surfaces to be made springy and to be moved about and bent at will. This gives another route for the eversion, as explained in the movie *Outside In*, released in 1994 by the Geometry Center and directed by the authors and Tamara Munzner. It can also be applied to other more general situations.

Outside In uses nontechnical language and computer animation to illustrate the process and to explain the concepts involved to a nonmathematical audience. Yet the video retains mathematical depth: we introduce the concept of a "regular homotopy" from topology, which is traditionally not encountered until advanced undergraduate mathematics classes. (Topology is the study of the properties of objects that remain unchanged even if the surface can be deformed, and differential topology is the study of smooth objects without corners or bends.) The metaphor we use is that of a material that can stretch and pass through itself, but that self–destructs if punctured or even pinched sharply. Of course, there is no such material in real life! That's where computer graphics comes in.

As one can see in Figure 1, the evolving sphere is highly symmetric; it is made up of sixteen congruent pieces (eight on top, eight on the bottom). The actual number of pieces is not very important, except for esthetic purposes; it is the fact that the sphere can be decomposed into these pieces at all, and in a symmetric way, that makes this method relatively easy to understand.

Still, it is a challenge to explain the process in a way that can be grasped in its entirety. A complete mental picture of any eversion is very hard to keep in mind, and clear printed or even animated images are hard to draw, because of the many layers of surface: the most successful expositions of the everting sphere concentrate on local details, and rely on the viewer's powers of synthesis to piece the details together. Symmetry is an excellent way to concentrate on just one piece—the local picture—rather than trying to visualize everything simultaneously. So that's the route that *Outside In* takes.

Structure of *Outside In*

The framework of the video is a dialogue between a female teacher and a male student. In the first scene they work out between themselves the ground rules of what it means to turn a sphere inside out. The surface can stretch and bend, and pass through itself, but cannot be ripped, punctured, or creased.

(This leads to the question, which we skirt in the movie for lack of time: why turn a sphere inside out, and why use these rules? The short answer is that these rules give a mathematical puzzle that is challenging and counterintuitive, and yet has a solution. But the rules also stem from a classification problem of surfaces and deformations, more specifically of "immersions", or smooth surfaces without kinks. In these immersions, the notion of "sidedness" is also encoded into the surfaces. This property of sidedness is preserved no matter which side of the sphere is inside or out.)

Because creases are not allowed, simply pushing the sphere's caps straight through one another won't work: a tight loop or crease will form around the equator. How can this work? The student remains skeptical that the problem can be solved under these rules. If anything his skepticism increases in subsequent scenes, as the teacher persuades him that a circle cannot be turned inside out under the same rules. However, an idea that is introduced in connection with curves—namely, adding waves, or corrugations—turns out to be useful for surfaces as well. The teacher builds up the ideas needed for the grand finale: a step–by–step demonstration of the sphere eversion, seen from many angles and in many different cross sections.

The Uses of Symmetry

Although the process looks quite complex at first glance, it can be decomposed into relatively simple pieces both in space and in time. Spatially, we divide the sphere into eight strips that go from pole to pole, and we can further cut this strip crosswise into two equal pieces. (Imagine a peeled orange with eight segments, then keep only one segment and slice it through at the "equator".) Everything that happens to one half–strip happens at the same time to the others; that is, the eversion is symmetrical, as already discussed. This one–sixteenth is the fundamental building block of the eversion—–the entire sphere can be created by rotating copies of this piece (see Figure 2).

Temporally, the eversion is divided into five stages. First, we corrugate the sphere, that is, we create bulges in the middle of each of the strips. Second, we push the poles through each other, stopping when we create a loop at the equator. Third, we twist the two caps in opposite directions to convert the loops in the middle to twisting at the caps. (This is similar to taking a belt with a loop in it and pulling the ends tight: the loops can turn into twists.) This is the least transparent part of the eversion; the crucial fact is that the symmetrical corrugations we have introduced make the surface very pliable, so the twisting can be carried out without introducing creases. Fourth, we push the equatorial region of the surface radially through the center. Finally, we uncorrugate to complete the eversion.

Note that, in addition to the spatial symmetry, there is a type of time symmetry, or rather duality, as well. If we were to run the movie backwards, the first stage would also be one of corrugation, only the sphere would bulge in, instead of out. The second stage would be one of exchanging opposite ends, only the ends that are exchanged are opposite points on the equator, rather than the north and south pole. The third stage is dual to itself.

In Conclusion

One of our main goals in making *Outside In* is to demonstrate that seeing it rather than reading about it enables comprehension. It is at this point where words are inadequate and the video must be shown!

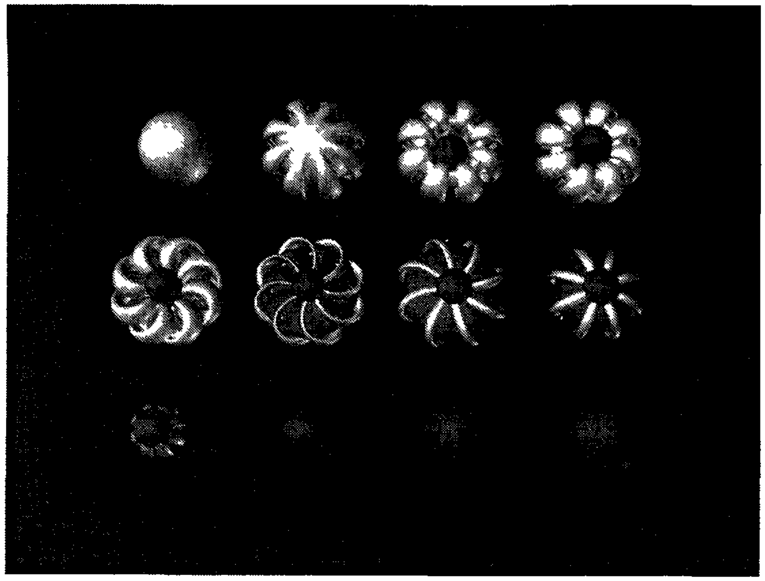

Figure 1

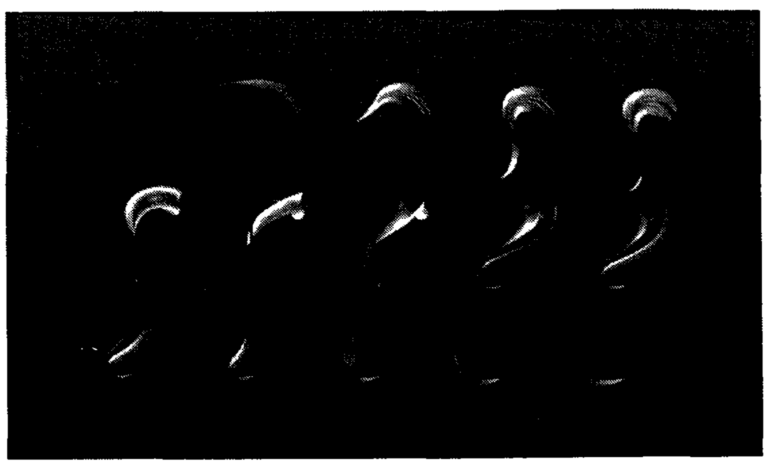

Figure 2