Symmetry and Density of Ellipse Regular Packings

Masaharu TANEMURA (Inst. Statist. Math.)
Takeo MATSUMOTO (Kanazawa Univ.)

Abstract

For the system of periodic pattern of identical ellipses, there are seven plane groups for which each ellipse has six contacting neighbours. For most of the seven groups, the computation of packing density is not a trivial task. In this paper, recent works by the present authors are reviewed and some discussions about the relation between the symmetry properties of density and the plane group symmetries.

Keywords: Ellipse Packing, Plane Group Symmetry, Packing Density, REDUCE

1 Introduction

A pattern of non-overlapping objects has long attracted many people especially when the objects are circles or spheres. One of the interesting questions for the pattern was "how dense the objects are packed?" In case of the system of identical circles, it was Thue (1910) who first proved that the maximum density of circles is $\pi/\sqrt{12}$ and that the maximum density is attained for the hexagonal packing among any competitive configurations of circles. On the other hand, the problem of densest packing of spheres, known as a conjecture by Kepler (1619), is still unsolved for the present.

For the system of identical ellipses, however, it is just fifty years since Nowacki (1948) had considered the four plane group symmetries, *p2*, *c2mm*, *p2gg* and *p31m*, which would have high density and in which every ellipse is in contact with six neighbouring ellipses. And it is about ten years ago that Grünbaum and Shephard (1987) have pointed out that there exist further three packing systems, i.e., *p3* and two kinds of *p2gg-4c1*, in which every ellipse has six contacting neighbours.

As regards the packing density of ellipses, Matsumoto and Nowacki (1966) proved that *p2* and *c2mm* packings of ellipses always attain the densest packing density $\pi/\sqrt{12}$ of circles while Matsumoto (1968) has shown that this density can be never attained for the *p2gg* packing of ellipses.

For the rest four types of packing of ellipses, however, the equations become too complicated to handle by hand calculations in order to compute the density in each of ellipse packing.

We have recently presented the proof, with the aid of computer, that the other four packings of ellipses, too, cannot exceed the maximum density of circles. In this paper, we review our recent works and give some discussions about the relation between the symmetry properties of density and the plane group symmetries.

2 *p31m* and *p3* packings of ellipses

As regards the system *p31m*, Tanemura and Matsumoto (1992) computed the packing densities by the use of mathematical software REDUCE and the use of numerical computational programs, and proved that it attains the maximum density $\rho = \pi/\sqrt{12}$ only when the aspect ratio $k \equiv a/b$ of the ellipse major axis a and minor axis b becomes unity.

Another system of densest ellipse packing, *p3*, was investigated by Matsumoto and Tanemura (1995) in a similar manner with the aid of computer, and the density attains the maximum $\pi/(\sqrt{12})$ only when the aspect ratio $k = 1$. In this case, however, different from the case of the system of *p31m*, the tilt angle θ of ellipse major axis becomes a variable, then the treatment of equations tends to be labourious.

It is known that the unit cell of the *p3* packing of ellipses is a rhombus and that it includes three ellipses as shown in Fig. 1. In order to describe the *p3* packing of ellipses completely, it is shown that nine unknown variables should be introduced (Matsumoto & Tanemura, 1995). Then, we can derive nine equations (simultaneous nonlinear equations) for the unknown variables by using the conditions of contactness among ellipses. In these equations there are two parameters, namely k and θ, where $k = a/b$ is the aspect ratio of the ellipse ($2a$ and $2b$ are the usual major and minor axes of ellipse, respectively), and where θ is the tilt angle of major axis.

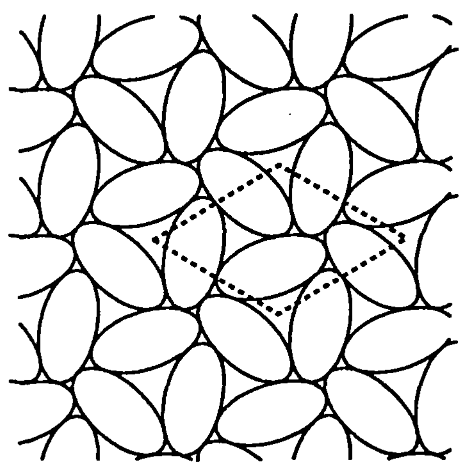

Fig.1 Sample configuration of *p3* packing of ellipses.
A unit cell is shown by a rhombus with broken lines.

To solve the set of equations, we used a numerical software. Once the set of variables is known for each of parameter values (k, θ), it is easy to calculate the density $\rho(k, \theta)$ of the *p3* packing of ellipses as a function of (k, θ). From the plot of $\rho(k, \theta)$, we observe that $\rho(k, \theta)$ never exceeds the value $\pi/\sqrt{12} = 0.906899682117 \cdots$.

The last statement can be proven, near $k = 1$, by expanding $\rho(k, \theta)$ in terms of $\varepsilon \equiv k - 1$. By employing the software REDUCE, we obtain the following expansion form for $\rho(k, \theta)$ up to the term $O(\varepsilon^4)$:

$$\begin{aligned} \rho(k,\theta) &= \frac{\pi}{\sqrt{12}}\left\{1 - \frac{3}{32}\varepsilon^2 \right. \\ &\quad + \frac{1}{128}(17 - 90\sin^2\theta + 240\sin^4\theta - 160\sin^6\theta)\varepsilon^3 \\ &\quad \left. - \frac{3}{4096}(411 - 1440\sin^2\theta + 3840\sin^4\theta - 2560\sin^6\theta)\varepsilon^4 \right\} + O(\varepsilon^5). \end{aligned} \tag{1}$$

This expression indicates that $\rho(k, \theta)$ shows a maximum at $k = 1$ and that $\rho(1, \theta) = \pi/\sqrt{12}$ holds for any θ.

3 *p2gg-4c1* packings

There are two types of *p2gg-4c1* packing of ellipses as shown in Figs. 2 and 3. To distinguish the two types, we call them *p2gg-4c1*(a) and *p2gg-4c1*(b) correspondingly to Figs. 2 and 3.

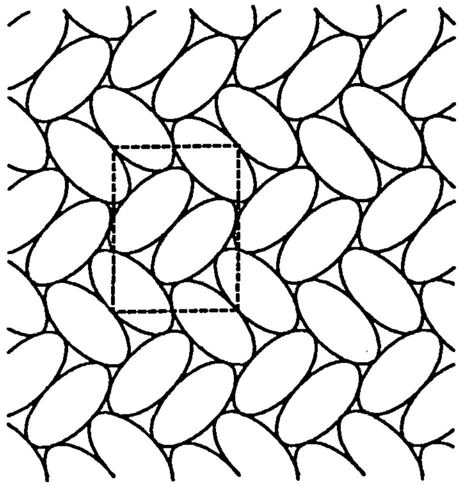

Fig.2 Sample configuration of *p2gg-4c1*(a) packing of ellipses. A unit cell is indicated by a rectangular with broken lines.

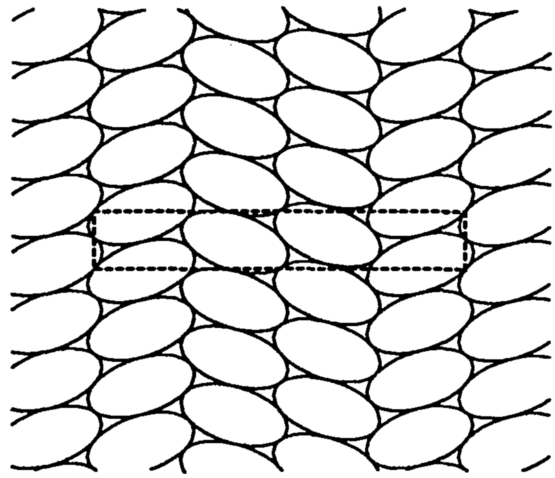

Fig.3 Sample configuration of *p2gg-4c1*(b) packing of ellipses. A unit cell is indicated by a rectangular with broken lines.

In order to describe the *p2gg-4c1*(a) packing of ellpses, it is shown that eight unknown variables are necessary (Tanemura & Matsumoto, 1997). For the *p2gg-4c1*(b) packing, on the other hand, six unknown variables are sufficient (Tanemura & Matsumoto, 1998).

By performing the numerical computations, we can get the packing density $\rho(k,\theta)$ as a function of parameter (k,θ) for each of *p2gg-4c1* packings. We made the plots of density of the *p2gg-4c1*(a) and *p2gg-4c1*(b) and observed that the density $\rho(k,\theta)$ of both of *p2gg-4c1* packings of ellipses never exceeds $\pi/\sqrt{12}$.

We can again prove the above statement by employing the series expansion of $\rho(k,\theta)$ in terms of $\varepsilon = k-1$. The result for the *p2gg-4c1*(a) packing is

$$\begin{aligned}\rho(k,\theta) &= \frac{\pi}{\sqrt{12}}\{1-\frac{5}{3}\sin^2 2\theta\cdot\varepsilon^2+\frac{5}{3}\sin^2 2\theta\cdot\varepsilon^3-\frac{5}{12}\sin^2 2\theta\,(5-6\sin^2 2\theta)\cdot\varepsilon^4\\ &+\frac{5}{2}\sin^2 2\theta\,(1-2\sin^2 2\theta)\cdot\varepsilon^5\}+O(\varepsilon^6),\end{aligned} \tag{2}$$

whereas for the *p2gg-4c1*(b) packing, the result is

$$\begin{aligned}\rho(k,\theta) &= \frac{\pi}{\sqrt{12}}\{1-\frac{1}{48}\sin^2 2\theta\cdot\varepsilon^2+\frac{1}{48}\sin^2 2\theta\cdot\varepsilon^3\\ &-\frac{5}{144}\sin^2\theta\,(3-10\sin^2\theta+14\sin^4\theta-7\sin^6\theta)\cdot\varepsilon^4\\ &+\frac{1}{72}\sin^2\theta\,(9-44\sin^2\theta+70\sin^4\theta-35\sin^6\theta)\cdot\varepsilon^5\}+O(\varepsilon^6).\end{aligned} \tag{3}$$

These equations indicate that the statement 'the density $\rho(k,\theta)$ of *p2gg-4c1* packings of ellipses never exceeds $\pi/\sqrt{12}$' is true.

4 Discussion

We observed, from the plots of density $\rho(k,\theta)$, that there exist some symmetry properties of the density $\rho(k,\theta)$ in the plane (k,θ). In case of the *p3* packing, the symmetry relations of density derived from the symmetry of the plane group *p3* are summarized as

$$\rho(k,\theta) = \rho(k,-\theta) = \rho(k,\theta+\pi/3) = \rho(1/k,\ \theta+\pi/6). \tag{4}$$

These properties are in accord with the results of our numerical computations for the *p3* packing of ellipses. Expansion form for $\rho(k,\theta)$ in Eq.(1) also satisfy the above properties (4).

For the case of *p2gg-4c1* packings of ellipses, the symmetry relations derive from the plane group symmetry are given as

$$\rho(k,\theta) = \rho(k,-\theta) = \rho(k,\theta+\pi) = \rho(1/k,\ \theta+\pi/3). \tag{5}$$

The symmetry properties of density derived by our numerical computations, however, are summarized as

$$\rho(k,\theta) = \rho(1/k,\ \theta) = \rho(k,\theta+\pi/2) \tag{6}$$

in addition to the above properties (5). This fact means that, for example, for a given set of values (k_0,θ_0), the density of a *p2gg-4c1* packing corresponding to (k_0,θ_0) is equal to that corresponding to $(k_0,\theta_0+\pi/2)$, although the shapes of unit cell are generally different with each other. It should be finally pointed out that the expansion forms of $\rho(k,\theta)$ given in Eqs. (2) and (3) also satisfy all of the symmetry properties of density derived from the plane group symmetry (5) and those derived by our numerical computations (6).

References

Grünbaum, B. and Shephard, G.C. (1987). *Tilings and Patterns*, p.700. Freeman, New York.

Kepler, J. (1619). *Harmonice Mundi.* Lincii. [German translation: *Johannes Kepler Gesammelte Werke*, M.Caspar, ed. Band VI. Beck, Munich, 1940.]

Matsumoto, T. (1968). Proof that the *pgg* packing of ellipses has never the maximum density. *Zeitschrift für Kristallographie*, **126**, 170–174.

Matsumoto, T. and Nowacki, W. (1966). On densest packing of ellipsoids. *Zeitschrift für Kristallographie*, **123**, 401–421.

Matsumoto, T. and Tanemura, M. (1995). Density of the *p3* packing of ellipses. *Zeitschrift für Kristallographie*, **210**, 585–596.

Nowacki, W. (1948). Symmetrie und physikalisch-chemische Eigenschaften kristallisierter Verbindungen. V. Über Ellipsenpackungen in der Kristallebene. *Schweiz. Mineralog. Petrog. Mitt.*, **28**, 502–508.

Tanemura, M. and Matsumoto, T. (1992). On the density of *p31m* packing of ellipses. *Zeitschrift für Kristallographie*, **198**, 89–99.

Tanemura, M. and Matsumoto, T. (1997). Density of the *p2gg-4c1* packing of ellipses (I). *Zeitschrift für Kristallographie*, **212**, 637–647.

Tanemura, M. and Matsumoto, T. (1998). Density of the *p2gg-4c1* packing of ellipses (II). *in preparation.*

Thue, A. (1910) Über die dichteste Zusammenstellung von kongruenten Kreisen in einer Ebene. *Christiania Vid. Selsk. Skr.*, **1**, 3–9.

Author for Correspondence:

Full Name: Masaharu Tanemura
Full Address: The Institute of Statistical Mathematics,
4-6-7 Minami-Azabu, Minato-ku, Tokyo 106-8569, Japan
E-mail: tanemura@ism.ac.jp