RIDDLED AND INTERMINGLED BASINS

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It has become apparent over the past decades that the long-term behavior of even the most elementary-looking dynamical systems on Euclidean spaces or manifolds can be extremely complicated, even if attention is restricted to "typical" systems. If there is some underlying geometric restriction on the system, the notion of a typical system changes, and behavior particular to the restricted class of system is in a sense also typical. For example, if the system is Hamiltonian (volume preserving), KAM tori and Arnold diffusion are likely. If the system is invariant under some group of symmetries, symmetry breaking bifurcations are typical. We consider systems which are geometrically restricted in another way. The simplest nontrivial case is a system $F : \mathbf{R}^2 \to \mathbf{R}^2$ of the plane for which there is a line N such that $F(N) \subset N$. The line N is called invariant under F and we may consider the restriction of F to N, denoted $F_N : N \to N$, which is itself a dynamical system on N. The restricted system F_N has its own dynamics—for example, it may be a simple quadratic logistic map and have a chaotic attractor A in N. We are interested in the role of A in the larger system F; for example, is A and attractor of some domain in $\mathbf{R}^2$?

More generally, $\mathbf{R}^2$ can be replaced by a manifold, and N can be replaced by a submanifold invariant under F. Such systems can arise naturally in models with constraints or symmetries [Chossat & Golubitsky, 1988; Fujisaka et al., 1986; King & Stewart, 1991; Pecora & Carroll, 1990].

The *positive limit set* $L^+(x)$ of a point x is the set of points y which are limit points of the trajectory $F^n(x)$ as $n \to \infty$. A set A is an *attractor* if a point x chosen at random (with respect to some uniform distribution) near A has a positive probability that $L^+(x) = A$. The *basin of attraction* $\beta(A)$ is the set of points $\{x : L^+(x) \subset A\}$. In most systems studies, a basin of attraction is typically an open set and in particular contains disks [McDonald et al., 1985]. The purpose of this talk is to exhibit systems with basins with the opposite property: they contain no disks. Indeed a point chosen at random from any disk has positive probability of being in $\beta(A)$ and a positive probability of not being in $\beta(A)$. The basin $\beta(A)$ is riddled with holes, and so we call it a *riddled basin.* Such behavior is robust under changes to the system; that is, it is typical. We exhibit examples, both numerical and mathematically rigorous.

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