

QUASISYMMETRY TYPES FOR ALMOST PERIODIC TILINGS

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A basic idea in the theory of periodic tilings is the classification into *symmetry types* corresponding to isomorphism classes of *symmetry groups*. Here we describe a natural way to generalize the theory of symmetry types to the more general class of *almost periodic tilings* using ideas from dynamical systems theory. A tiling x is said to be *almost periodic* if for every patch q in x there exists a positive number r such that a copy of q occurs within a distance r from an arbitrary location in x . Any periodic tiling is almost periodic, but there are also aperiodic almost periodic tilings. The best known examples of almost periodic tilings are the *Penrose tilings*; many new examples have been found in connection with the theory of quasicrystals.

Almost periodic tilings correspond to the almost periodic points in a type of group action associated with sets of tilings, called a *tiling dynamical system*. A fundamental theorem of W. Gottschalk shows that such actions are *minimal*, a property generalizing the *transitivity* which occurs in the periodic case. In particular, if x is a periodic tiling of $\mathbf{R}^n$, and $\mathbf{Z}_x$ is its translation group, then the quotient $\mathbf{R}^n/\mathbf{Z}_x$ is the n -dimensional torus corresponding to a fundamental domain of x . The translation action by $\mathbf{R}^n$ is transitive. There is also natural action of the point group $\mathbf{H}_x$ of x on $\mathbf{R}^n/\mathbf{Z}_x$. We call the semidirect product $\mathbf{G}_x$ of $\mathbf{R}^n$ and $\mathbf{H}_x$ the *quasisymmetry group* of x , and we show that one can recover the classical symmetry type of x by knowing the appropriate dynamical *isomorphism class* of the natural action of $\mathbf{G}_x$ on $\mathbf{R}^n/\mathbf{Z}_x$.

When we switch to the almost periodic case, we replace $\mathbf{R}^n/\mathbf{Z}_x$ with the *orbit closure*

$\overline{O(x)}$ (in a *tiling topology*) of the tiling x , under the action of $\mathbf{R}^n$ by translation. In this general setting the point group $\mathbf{H}_x$ is defined to be the set of rigid motions leaving $\overline{O(x)}$ invariant. In the aperiodic case, the action of $\mathbf{G}_x$ on $\overline{O(x)}$ is *minimal* rather than transitive. In analogy to the above, the *quasisymmetry type* of an almost periodic tiling is defined to be the dynamical isomorphism class of this action. This provides a classification which generalizes classical symmetry theory while retaining many of its most important features. The new theory includes a version of Bieberbach's second theorem (the rigidity of isomorphism between quasisymmetry groups); on the other hand, there is no crystallographic restriction for almost periodic tilings (hence their place in the theory of quasicrystals), and in several different ways, no generalization of Bieberbach's third theorem. There is, however, a close relation between the *discrete spectrum* of the tiling dynamical system corresponding to a tiling x and the x-ray diffraction analysis of almost periodic structures related to x . A few cases, including the Penrose tiling case, have been analyzed completely.