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POLYHEDRAL GEOMETRY AS A BASIS FOR BUILDING STRUCTURES AND OTHER MATERIALS SYSTEMS

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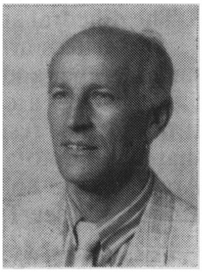
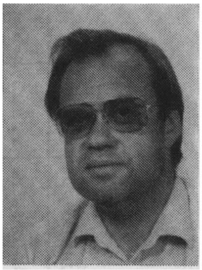
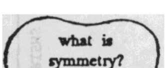




QUESTION 1
THE GEOMETRY OF UNIFORM POLYHEDRA AND THEIR DERIVATIVES



Introduction

Polyhedra are usually described as a set of polygons enclosing together a portion of space. This definition is well in accordance with what architecture aims at. As a matter of fact most building structures are in their general shape (macroform) or in their structural configuration (microform) based on polyhedral geometry. It seems that at this moment a revival of the apprehension is noticeable by architects and artists of the polyhedron as an important medium for the expression of their ideas. Most of our buildings are momentarily based on the shape of the cube or of the prism, notwithstanding the fact that many other forms exist, which have more potentials and are much more interesting. The cube and the prism are members of a large family of regular and semi-regular solids: the so-called uniform polyhedra, of which representatives are found in nature itself in the form of crystals, viruses and radiolaria. Bees adopted the hexagonal cell apparently as an economic solution for their housing problem.

Uniform polyhedra

The so-called uniform polyhedra are characterized by the following five limiting conditions:

1. They are composed of 1, 2 or 3 kinds of plane, regular polygons.
2. The polygons meet in pairs at a common edge.
3. The dihedral angle at such an edge is always convex, i.e. less than 180.
4. All vertices are identical so that at each vertex the same polygons occur in the same number and in the same order of sequence.
5. All vertices are positioned upon one circumscribed sphere.

If the further restriction is made, that the polygon has not more than 10 edges, in that case a total number of 30 different solids is found: 5 Platonic and 15 Archimedean solids (including 2 left and right-handed versions), 5 prisms and 5 antiprisms (see Table 1).

	Serial number P	NAME	Vertices	Edges	Faces	Vertex-symbol	Radius of circumsphere
PLATONIC	1	tetrahedron	4	6	4	3-3-3	0.6123724356958
	2	cube	8	12	6	4-4-4	0.8660254037844
	3	octahedron	6	12	8	3-3-3-3	0.7071067811865
	4	dodecahedron	20	30	12	5-5-5	1.4012585384441
	5	icosahedron	12	30	20	3-3-3-3-3	0.9510565162952
ARCHIMEDEAN	6	truncated tetrahedron	12	18	8	3-6-6	1.1726039399559
	7	cuboctahedron	12	24	14	3-4-3-4	1.0000000000000
	8	truncated octahedron	24	36	14	4-6-6	1.5811388300842
	9	truncated cube	24	36	14	3-8-8	1.7788236456639
	10	rhombicuboctahedron	24	48	26	3-4-4-4	1.3989663259659
	11	truncated cuboctahedron	48	72	26	4-6-8	2.3176109128928
	12	icosidodecahedron	30	60	32	3-5-3-5	1.6180339887499
	13	truncated icosahedron	60	90	32	5-6-6	2.4780186590676
	14	truncated dodecahedron	60	90	32	3-10-10	2.9694490158634
	15	snub cube	24	60	38	3-3-3-3-4	1.3437133737446
	16	rhombicosidodecahedron	60	120	62	3-4-5-4	2.2329505094157
	17	truncated icosidodecahedron	120	180	62	4-6-10	3.8023944998513
	18	snub dodecahedron	60	150	92	3-3-3-3-5	2.1558373751152
PRISMS	19	triangular prism	6	9	5	4-4-3	0.7637626158260
	20	pentagonal prism	10	15	7	4-4-5	0.9867151553260
	21	hexagonal prism	12	18	8	4-4-6	1.1180339887499
	22	octagonal prism	16	24	10	4-4-8	1.3989663259659
	23	decagonal prism	20	30	12	4-4-10	1.6935270853311
ANTIPRISMS	24	square antiprism	8	16	10	3-3-3-4	0.8226643880080
	25	pentagonal antiprism	10	20	12	3-3-3-5	0.9510565162952
	26	hexagonal antiprism	12	24	14	3-3-3-6	1.0876638735805
	27	octogonal antiprism	16	32	18	3-3-3-8	1.3735485807735
	28	decagonal antiprism	20	40	22	3-3-3-10	1.6745047437426

Table 1: Data of polyhedra.

Platonic polyhedra

The five regular solids (Fig. 1.1) each are composed of one kind of faces, namely either of triangles (tetrahedron, octahedron and icosahedron) or of squares (cube) or of pentagons (dodecahedron). They are also known as the Platonic solids because Plato was the first who mentioned them in his Timaeus (300 B.C.).

Following Plato, Kepler attributed to the five regular solids a certain emotional value, the tetahedron being associated with fire, the cube with the earth, the octahedron with the air, and the icosahedron with water. Of the dodecahedron he says: "Dodecaëdron vero relinquitur corpori celesti, habens eundem planorum numerum quem zodiacus celestics signorum." (The dodecahedron truly suggests the form of the firmament, having as many faces as there are signs in the zodiac.)

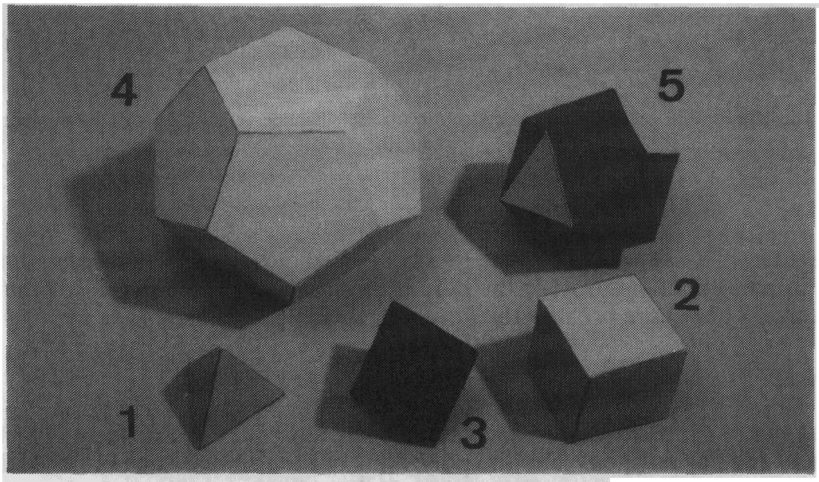


Figure 1.1: The 5 regular or Platonic solids.

Archimedean polyhedra

Another group having the same characteristics are the semiregular or Archimedean solids (in Heron's *Definitiones* their discovery is ascribed to Archimedes). They are made up of several (but not more than 3) kinds of polygons; in addition, for all these semi-regular solids the situation at each vertex is identical: the same polygons join each other in the same order, their greatest number being 5.

They share with the Platonic solids the property, that they each can be inscribed in a sphere that touches all its corners. Since they have more corners than the platonic solids, they provide closer approximation to the sphere.

Two representatives of this group, the snub cube and the dodecahedron, have a left and a right handed version; they are enantiomorphic. 4 solids have isomeric forms, which means that they can partially be differently arranged.

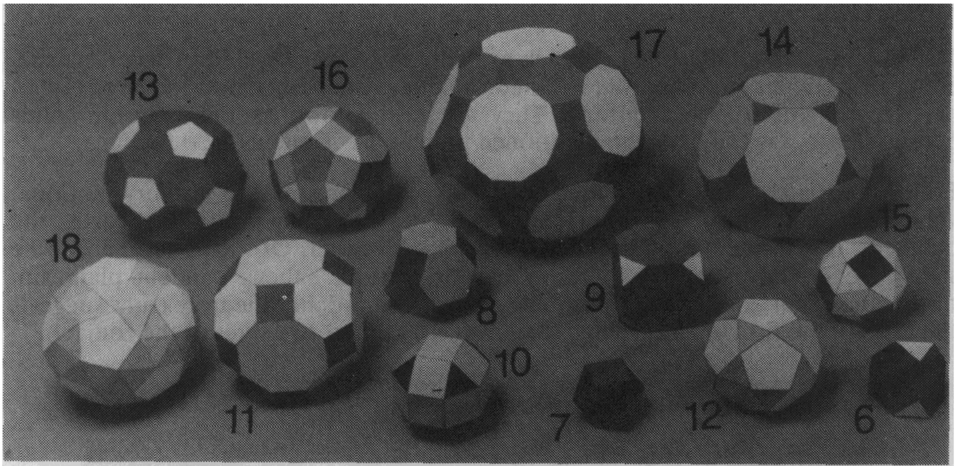


Figure 1.2: The 13 semi-regular or Archimedean solids. Left-handed versions exist of the numbers *P*15 and *P*18.

Prisms

The regular prisms, which consist of two congruent regular polygons similarly placed in parallel planes, satisfy the definition of facially regular solids if the sides are squares. The square prism is the familiar cube (Fig. 1.8).

Antiprisms

The regular antiprisms or prismoids have also parallel plane faces which are congruent regular polygons, but one polygon is twisted so that each of its vertices is midway between two vertices of the other. If the sides are equilateral triangles, it is facially regular (Fig. 1.9). The regular prisms and antiprisms are for this reason often called Archimedean. Like the other Archimedean solids, they can be inscribed in spheres and each has similar vertex situations. Since the number of regular polygons is endless, both series are actually unending. The triangular antiprism is the familiar octahedron and similarly the two-sided antiprism is identical to the tetrahedron.

Vertex figures

If one connects the other ends of the edges meeting at a vertex of a polyhedron, the vertex figure is found. It forms the basis of a pyramid having the original vertex as its apex. This pyramid contains all data, relevant for each polyhedron, i.e.

– edge length,
– circumferential angles of the meeting polygons,
– radius of curvature of the circumscribed sphere, as it is denied by the fact that the considered vertex as well as its neighbours all are positioned upon it,
– all occurring dihedral angles between polygons.

Any vertex pyramid is characteristic for a specific polyhedron. This can also be said of the vertex figures, of which three specimens occur: triangular, quadrangular and pentagonal – depending on the number of meeting polygons.

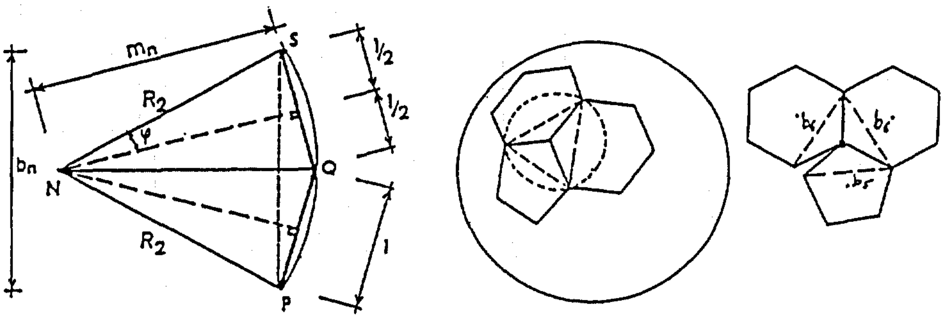


Figure 1.3: Relevant values of 2 adjacent polygons and schematic representation of polyhedron with code 5-6-6, showing vertex situation.

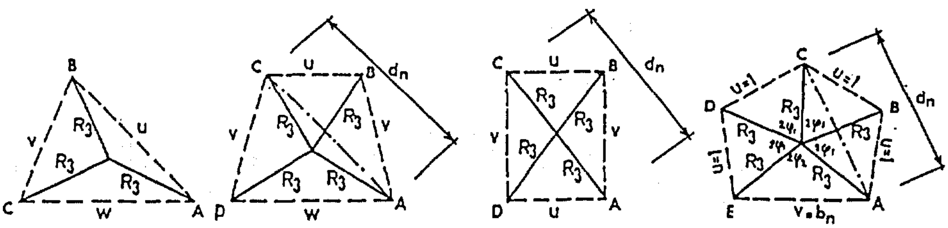


Figure 1.4: Different characteristic vertex figures.

Reciprocal solids

By connecting the bisectors of the meeting edges, a vertex figure of half the magnitude of the original is found. If these are extended until they intersect in pairs a new figure is obtained: the dual or reciprocal figure, which has as many faces – all of one kind – as the original polyhedron has vertices and vice versa (Fig. 1.6).

The platonic or regular solids appear to be reciprocal to each other, namely: dodecahedron-isocahedron, octahedron-cube, and tetrahedron-complementary tetrahedron (Fig. 1.5).

The reciprocals of the Archimedean or semi-regular solids have the characteristics:
– all faces are identical and have as many sides as the number of meeting polygons in the original solid;
– the number of the faces equals the original number of edges and vice versa;
– the number of edges remains the same;
– its edges bisect the original edges, tangent to the mid-sphere;
– all dihedral angles are mutually equal.

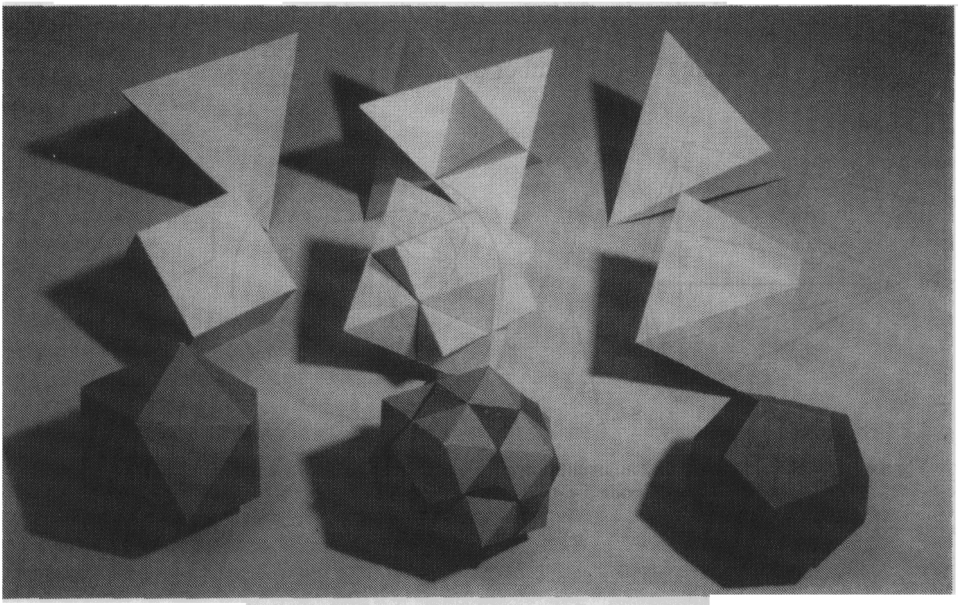


Figure 1.5: The reciprocity of the Platonic figures.

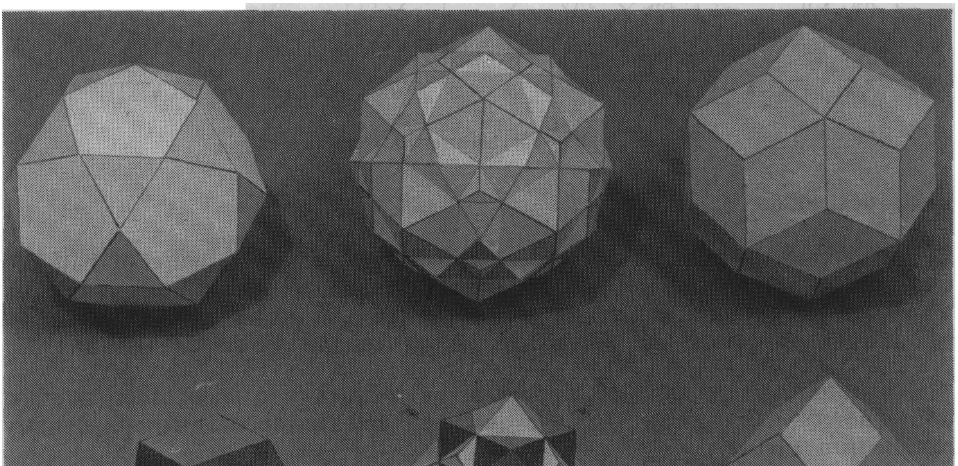


Figure 1.6: The principle of duality in uniform polyhedra, demonstrated for *P*7 and *P*12.

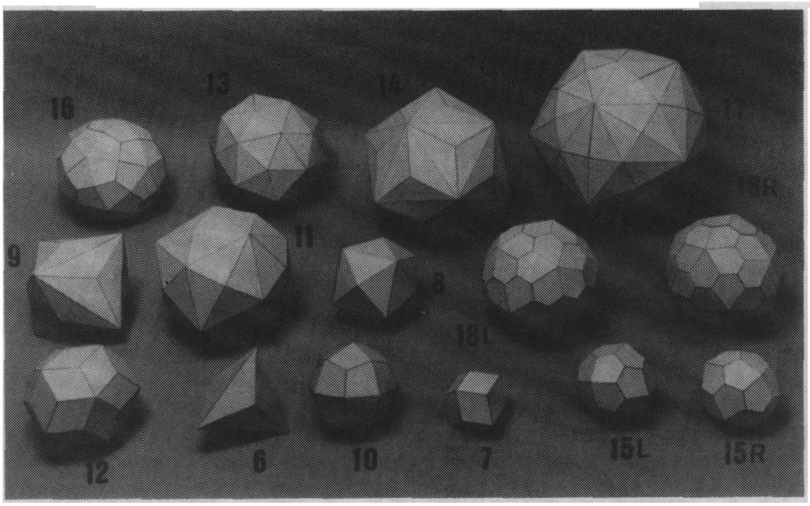


Figure 1.7: The reciprocal Archimedean figures.

Two famous representatives of this group are the rhomboid dodecahedron (honeycomb cell) and triacontahedron (Nos. 7 and 12).

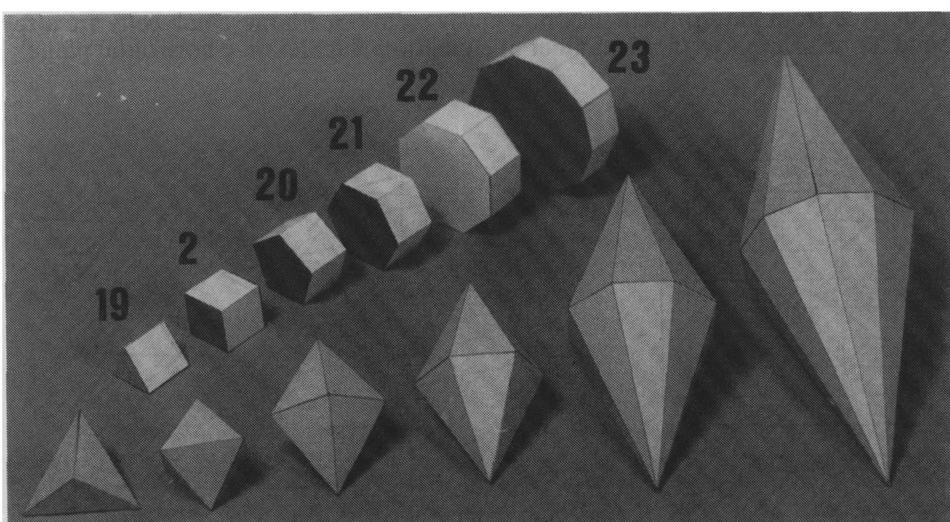


Figure 1.8: The prisms and their reciprocals.

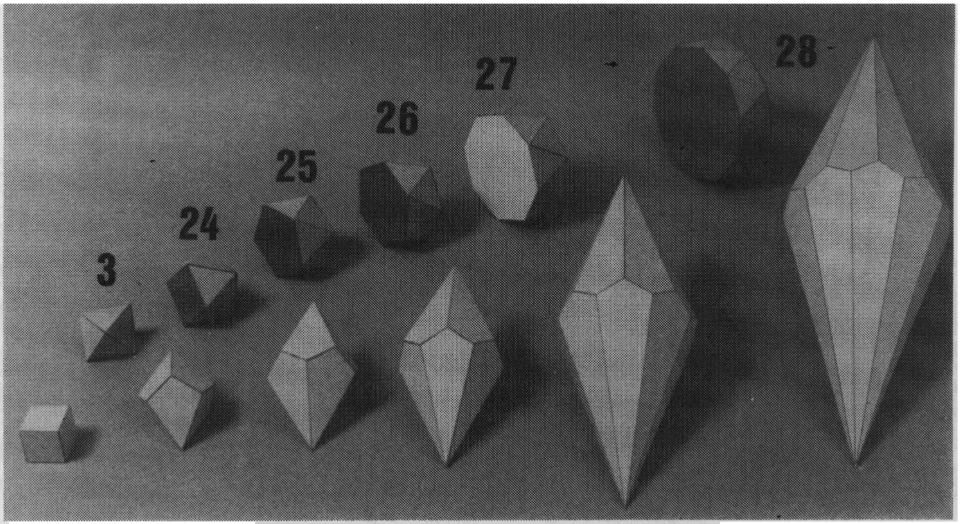


Figure 1.9: The antiprisms and their reciprocals.

Close-packings

The value of polyhedral solids for building structures lies in the fact that several can be stacked together, forming all-space filling packings (Fig. 1.10). Most of these are derived from nature itself. The previously mentioned honeycomb cells are also found if one brings together a number of equal soap bubbles on a horizontal plane.

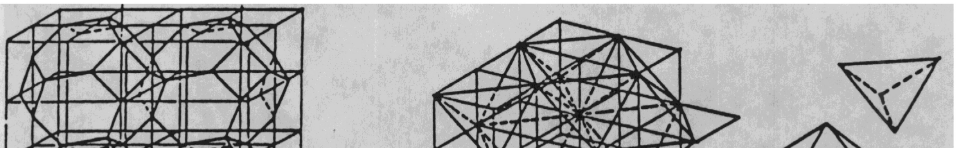


Figure 1.10: Various close-packings of polyhedra.

In the original honeycomb two of such layers are put back to back. This produces a close-packing of rhomboid dodecahedra, which have dihedral angles of 120°.

Critchlow discerns 1 regular packing, 4 semiregulars using 2 kinds and 5 using 3 kinds of polyhedra.

Starpolyhedra

If on the faces of a dodecahedron or of an icosahedron pentagonal, respectively triangular pyramids (12 or 20) are placed of such a height that their faces are in extension of the surrounding adjacent faces, star shaped solids are formed of great regularity: the so-called small- and large stellated dodecahedron. They were first mentioned by Johannes Kepler (1619) in his book *Harmonices Mundi.*

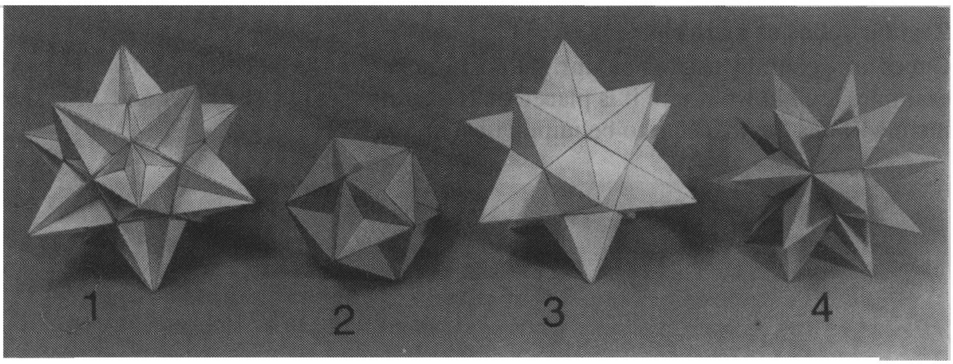


Figure 1.11: The 4 regular star polyhedra: 1) great icosahedron, 2) great dodecahedron, 3) small stellated dodecahedron, 4) great stellated dodecahedron.

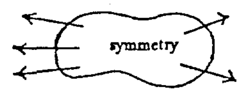
On closer inspection both these figures will be seen to be made up of 12 pentagrams, complexly interpenetrating.

Poinsot in 1809 discovered two more regular stellated polyhedra, the large dodecahedron and icosahedron (Fig. 1.11). To these M. Brückner (1900) added a great number of new – and often complicated – stellated figures. He also computed the principal geometrical data.

Conclusions

In recent years important contributions to our understanding of this matter have been made by, among other, Coxeter, Fejes Tóth, Lines, Roman, Cundy and Rollett, Critchlow, Wenninger, Holden, Williams, Pugh, each in his own field, often showing great interest in the application of such forms in art and architecture. Williams states: "It is possible to begin to develop a consciousness of form and more importantly a consciousness of the relationships among forms through form language". (Williams, 1972, p. 12). Certain shapes, such as cubes, prisms and pyramids, have regularly been employed, also in architecture, but it is generally felt that there are other forms, too, which offer at least equal possibilities for practical or artistic use. Up to now these other forms have been rather neglected, so to speak, by designers. Further study of these shapes may open up new vistas.

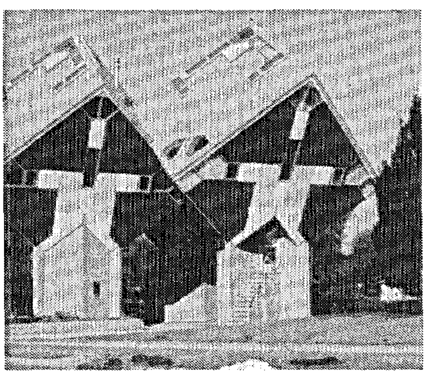
QUESTION 2
POLYHEDRAL BUILDING STRUCTURES



Although not many people are aware of it, most of the building structures around us are basically derived from polyhedra. Sometimes a cluster of such solids is formed, defining not only its overall shape but also its internal make-up. In the next chapter a brief impression is given of a number of applications. Most of the examples shown stem from the neighbourhood of the authors, but similar representers can in fact be found anywhere.

Regular solids as main element

Cubes are generally used in an upright position, but in the Netherlands two examples are known where it is placed on one of its vertices. (Figs. 2.1 and 2.2). The inclined walls do not seem to be unpractical in use.



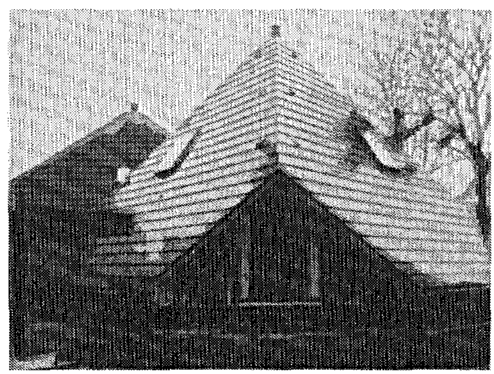


Figure 2.1: Cluster of tree-houses by P. Blom.
Figure 2.2: Facet-house by P. J. Gerssen at Gouderak (Veen, 1981).

In a substantial number of places, pyramidal houses were built being the upper half of an octahedron (Fig. 2.3). They are two stories high (Huybers, 1987).

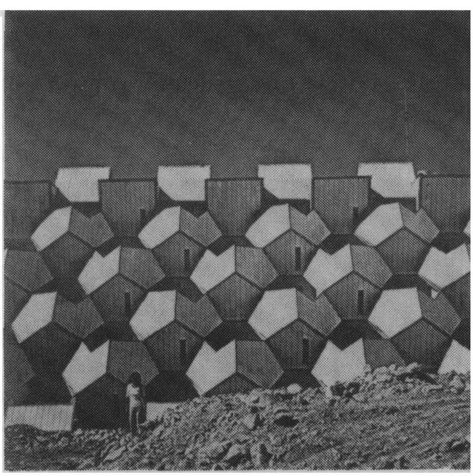
The dodecahedron is not very common, but it has been used for the so-called Do-Haus, designed by the group Polyteam of he University of Stuttgart and exhibited at Lüdenscheidt. In fact this has to be considered as a regular packing of cubes with roof-, and wall-elements of dodecahedral shape. Upon the faces of a dodecahedron a cube can be inscribed by drawing a specific set of diagonals. In the previous case pyramidal caps had been cut of from the dodecahedron.

Zvi Hecker designed the Ramot housing development near Jerusalem, basing the front of the building on a similar geometric arrangement of dodecahedra (Fig. 2.4).

The Dutch artist Gerard Caris designed a lot of dodecahedral objects, but also a housing system based on this shape (Fig. 2.5).

Figure 2.3: Pyramidal houses by G. Schouten at Huizen.

Recently an icosahedron-house has been designed and built at Almere-Haven (1982) by J. Abbo.



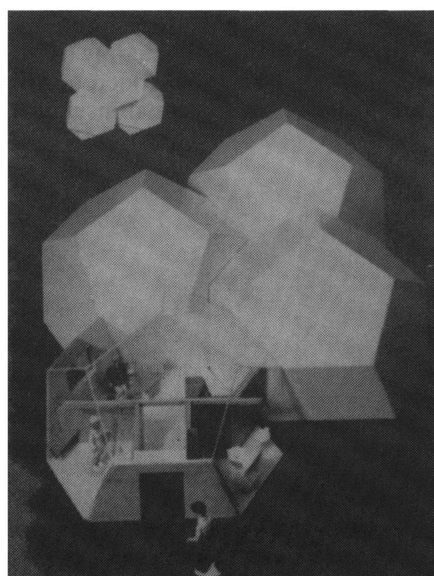
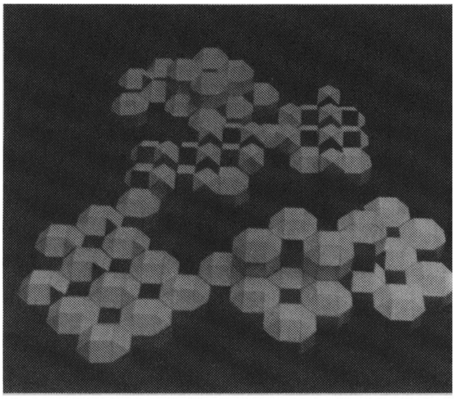
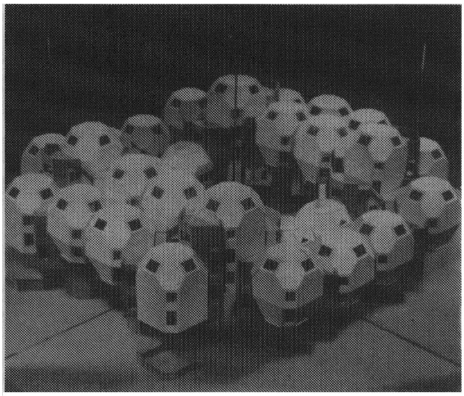


Figure 2.4. Ramot housing project by Zvi Hecker.
Figure 2.5. Dodecahedron house by G. Caris.

Semi regular solids

The formatting possibilities increase considerably if more than one kind of different polygons is allowed at the time. The rhombicuboctahedron or 'square spin' code 3-4-4-4, composed of 8 triangles and 18 squares had been applied by 3H-design (Hübner, Huster, and Hausermacher) for the construction of the sanitary cells at Munich (1972) of cardboard and sprayed-on GRP. It is known as the *Casanova-System* and has been used in 1976 a.o. at Neckarten-Zlingen for a more complex building.





Figures 2.6 and 2.7: Projects by students of Academy of Architecture at Amsterdam, J. Iserief and R. Dunnewolt, based on the rhombicuboctahedron and the truncated cuboctahedron.

The rhombicuboctahedron had also been adopted for the construction of an office at Mali, Africa. It consists of three octagonal units, of which the load-bearing structure has been made mainly of the termite resistant wood of the Rhonier palm. The walls are of hollow concrete bricks and the roofing of corrugated galvanized steel sheets with a grass-mat insulation at the back.

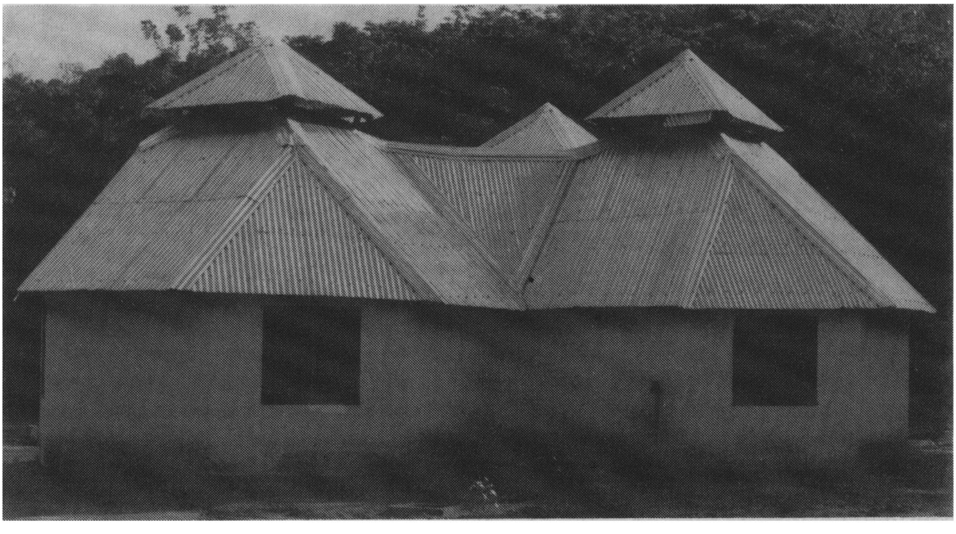
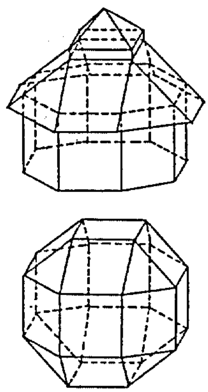


Figure 2.8: Office building at Bamako, Mali, consisting of three rhombicuboctahedra or 'square spins'.



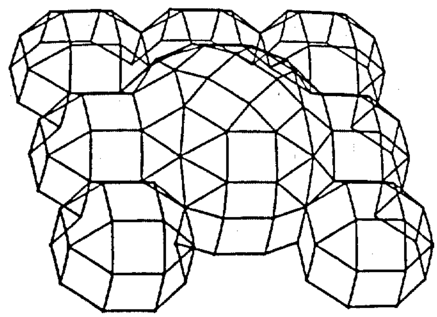


Figure 2.9: Scheme of office unit at Bamako.
Figure 2.10: Student project using 7 square spins and 1 truncated cuboctahedron (p11).

Attaching a series of antiprisms according their polygonal faces delivers a type of structure which is well-known and which has often been used. Versions of such structures in GRP have been built by B. S. Benjamin and R. Piano.

Antiprismatic folding structures have a concertina-like appearance and in fact they also behave that way. Models made of creased paper can be folded flat but they can also be folded in the direction of the axis of the circumscribed cylinder to form tight packages.

The shape of the structure is defined by that of the basic triangular element and by the dihedral angle between two adjacent triangles. They determine together the overall curvature. The relation of these 3 variables was thoroughly studied by Bleker (1970).

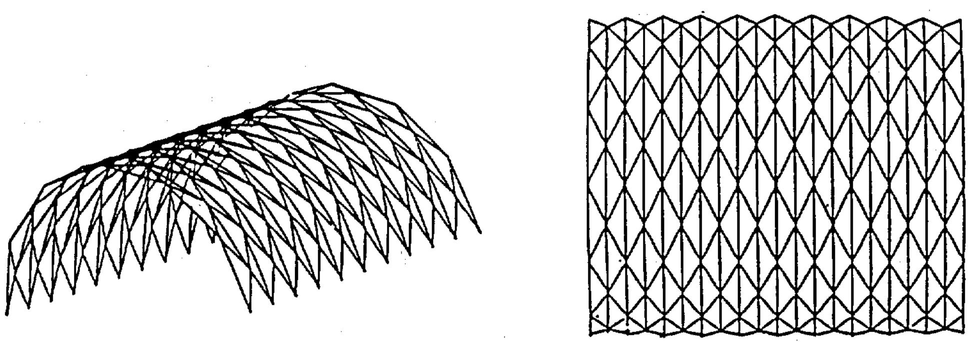


Figure 2.11: Antiprismatic folding structure.

Many others have added new information to the existing knowledge, a.o.:
– Árpád Kolozsváry (Universal Folded Plate System, Canada).
– Koryo Miura (Pseudo-cylindrical concave polyhedral shells, Vienna, 1970).
– Vinzenz Sedlak (Folded cardboard vaults, Guildford, 1975).

– Michael Kozel (Semi-regular deformations, Stuttgart, 1975).
– Peter Frostick (Form possibilities of antiprismatic surface active structures, Australia, 1977).

Apart from these, mention has to be made of Osvaldo L. Tonon (Argentina), A. Quarmby (England), Jörg Odebrett (Berlin), The Panelarch aluminium building system (Canada) and the GRP-system (Great Britain).

The smallest occurring ratio of edge length and radius of circumscribed sphere is 0.268, which means that a span of ultimately 7.5 times the edge length of the elements can be reached.

Thus for larger structures consequently larger panels would be necessary.

The 8- and 10-edged polygons are relatively large in comparison to the others and they are therefore not very interesting for structural reasons, but with the four remaining kinds (viz., triangular, square, pentagonal, and hexagonal) a considerable number of completely different polyhedra can be composed.

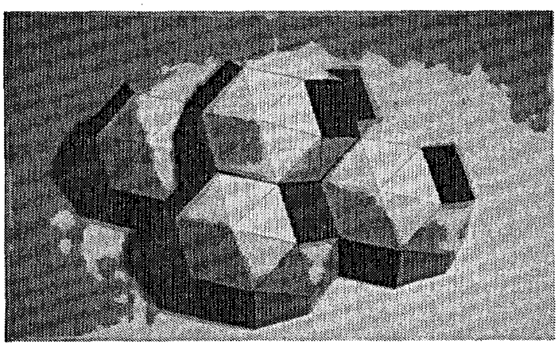


Figure 2.12: Cluster of pyramidized truncated octahedra.

A subdivision of the faces of the Platonic and Archimedean solids is obtained by triangulating the polygons. The centres of the polygons can be projected onto the circumscribed sphere and the newly found points are connected to the original polygon corners (Fig. 2.12). For the larger polygons – with 8 and 10 edges – this procedure does not result in a satisfactory solution, as still comparatively long inclined edges are obtained.

A better solution may be found in caps, reminding of that of the snub cube and of the rhombicuboctahedron (Fig. 2.10).

A further subdivision is found by a method which is usually referred to as geodesic subdivision of spherical surfaces (R. Buckminster Fuller).

In general three basically different geometrical arrangements are discerned (J. D. Clinton, 1971).

Class 1, icosahedron, or alternate ($K1$)
Class 2, rhomboid triacontahedron ($K2$)
Class 3, octahedron ($K3$)

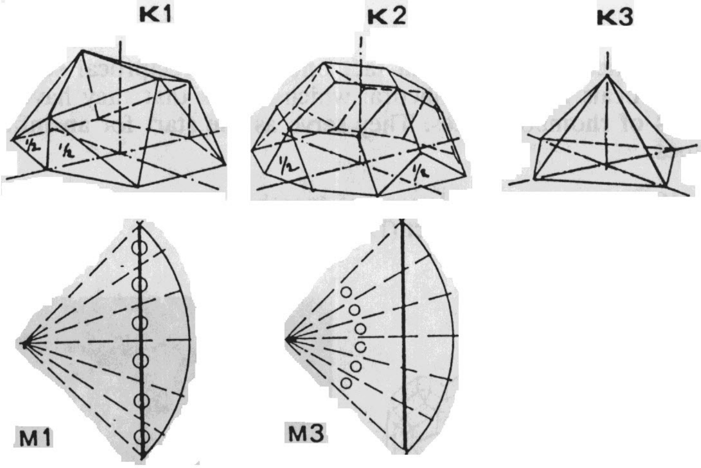


Figure 2.13: Triangulation of spherical surfaces.

The desired frequency is obtained by segmentation either of the original icosahedron edge or of its chord and by successive projection onto the sphere.

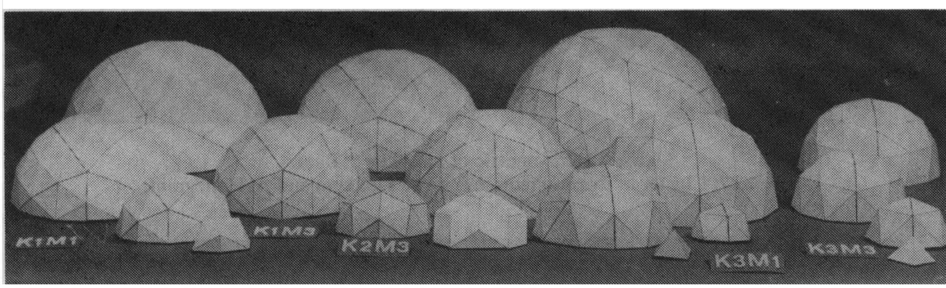


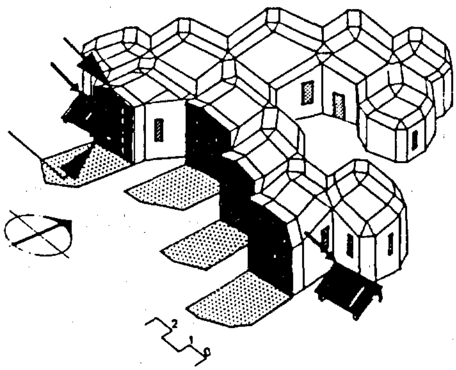
Figure 2.14: Models of geodesic dome structures of various classes and of different frequencies.

Figure 2.15: House consisting of truncated rhomboid dodecahedra by Steve Baer.

Isozonohedra

The rhomboid triacontahedron is in fact one of the reciprocal solids and both rhomboid reciprocals are isozonohedra: which means that they have continuous helicoidal rows of rhomboid faces. They serve as the start for an infinite row of possible shapes.

Steve Baer (1973), and Anton Hanegraaf together with Onno van Dokkum did interesting suggestions in this respect (Domebook, 1971)



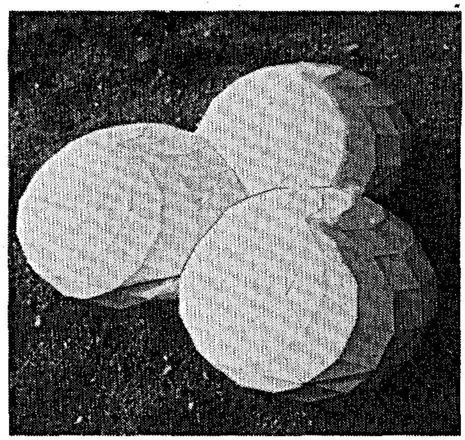


Figure 2.16: Isometric of house by S. Baer.
Figure 2.17: Cluster of isozonohedra by O. van Dokkum and A. Hanegraaf.

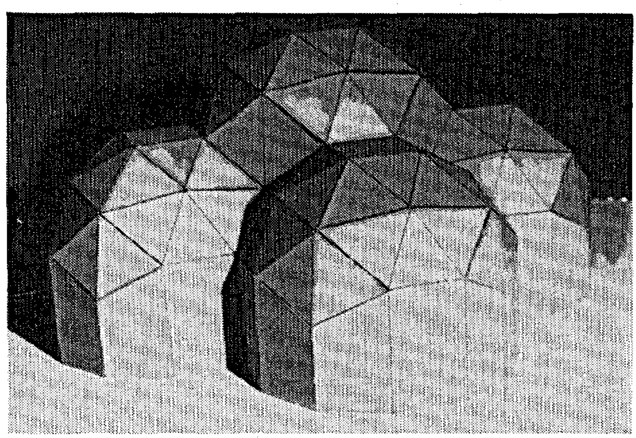


Figure 2.18: Octagon house by K. Critchlow.

K. Chritchlow suggested several building shapes based on close-packings of polyhedra. He constructed the so-called 'Octagon-house' from triple corrugated fibre board with a flame resistant glass-reinforced coating at both sides. The structure is a cluster of four basic units in close packing with one crowning unit (Fig. 18). It is based on the truncated rhomboid dodecahedron in a somewhat distorted version and with pyramids on the – irregular – hexagons.

Most of the space grid systems, that have been used in various places, can be considered as close-packings of 'immaterial' regular polyhedra.

This means that regular space patterns are formed showing a great uniformity and having a great versatility with the use of only a few different kinds of struts – sometimes even not more than one kind (Bleker, 1970).

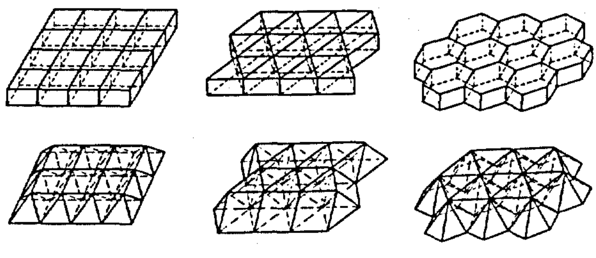


Figure 2.19: Various kinds of space structures consisting of regular elementary shapes.

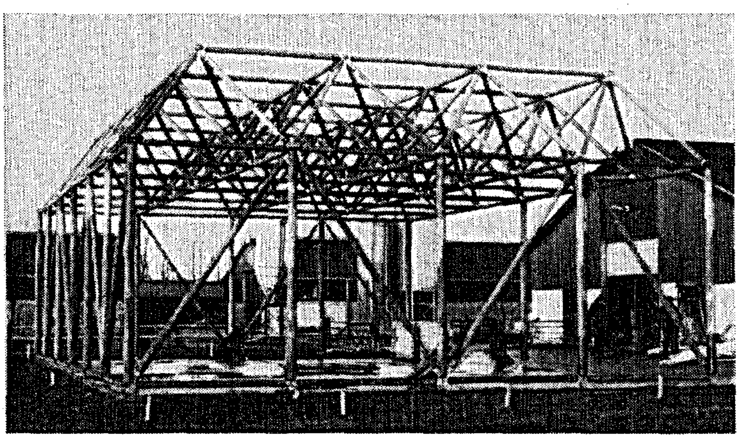


Figure 2.20: Example of a space structure of roundwood with a roof deck out of identical struts and having a configuration based on tetrahedra and half—octahedra.

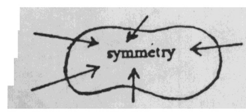
Conclusive remarks

This review is based on the authors' own experiences but it can be easily extended with a number of other examples from all over the world.

It is meant to give an impression of the various form-giving possibilities of the family of uniform solids.

Figure 2.21: Models of building shapes derived from the uniform polyhedra.

QUESTION 3
THE VISUALIZATION OF POLYHEDRAL SHAPES WITH COMPUTER TECHNIQUES



During the past 10 or 20 years many trials were undertoken to visualize building structures. The studies of Dragomir and Gheorghiu (Dragomir and Gheorghiu, 1978) and of Pál (1974) are magnificent examples of such approaches, at that time mainly based on – often elaborate – manual or algebraic methods.

The latter shows for instance a number of so called anaglyphs: combined red and green perspective sketches that – if observed through an appropriate pair of red and green glasses – give a marvellous three-dimensional view. Nooshin (1979) developed the Formex method with which many types of space structures can be envisaged with the help of a computer. This method has already shown great potentials in many cases.

The first author of this paper directed his attention since a long time on the geometry of polyhedra and of polyhedron-based structural shapes (Huybers, 1979a; 1980b; 1985; Huybers and Arends 1985; 1987). It appears that a great number of the known configurations can be described from this point of view. This was done also with the help of computer techniques. He worked together for some time with G. J. Arends (Arends and Huybers 1980; Huybers and Arends 1985; 1987) and recently with the second author G. v.d. Ende, who both had an essential part in the development of a program for personal computers (in this case written in GFA-Basic for Atari). It is not meant for commercial use but for study purposes only.

The interrelation of the uniform polyhedra

A direct relation exists between the 5 regular uniform polyhedra. This is demonstrated in Figure 3.1, where all four others are inscribed in the cube.

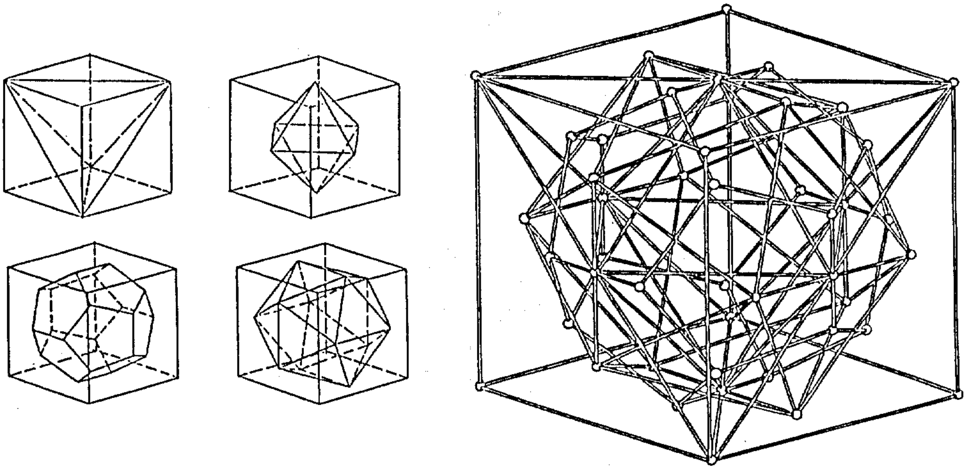


Figure 3.1: The relations between the Platonic solids.

As the semi-regular solids are derived from the regular solids by truncation of the vertices or of the edges, they as well can be found by derivation from the others. This truncation is done in such a way that new regular polygons are formed. The prisms and antiprisms can directly be derived from the cube.

Generating polyhedra

Therefore the uniform polyhedra can be said to consist of a set of regular polygons that are grouped in 3-dimensional space around a centre of gravity according specific rules. They can therefore be generated by firstly forming one of the polygons of the sometimes various kinds out of which a polyhedron consists and then placing it in space under certain angles, with respect to a given coordinate system, as many times as it occurs in the considered polygon (Huybers, 1979a; 1980b).

This is a combined translation and rotation procedure and it has to be repeated for each different kind of polygon.

It is done in a number of phases (Fig. 3.2):
1. A polygon with n edges (1 to 50) is formed and placed in the XY-plane with the origin as its centre (position F).
2. The n-gon is translated in positive direction along the Z-axis with the distance dz (position F').

3. If necessary a first rotation with the angle around the *Z*-axis is given to the polygon (position *F*").
4. Then a rotation around the *X*-axis is carried out (position *F*"').
5. At last the polygon is rotated with the angle around the *Y*-axis (position *F*"").
6. This is done as many times as the number of each kind of polygons that occurs.
7. The whole procedure is repeated for every kind of polygon (Huybers and Arends, 1985).

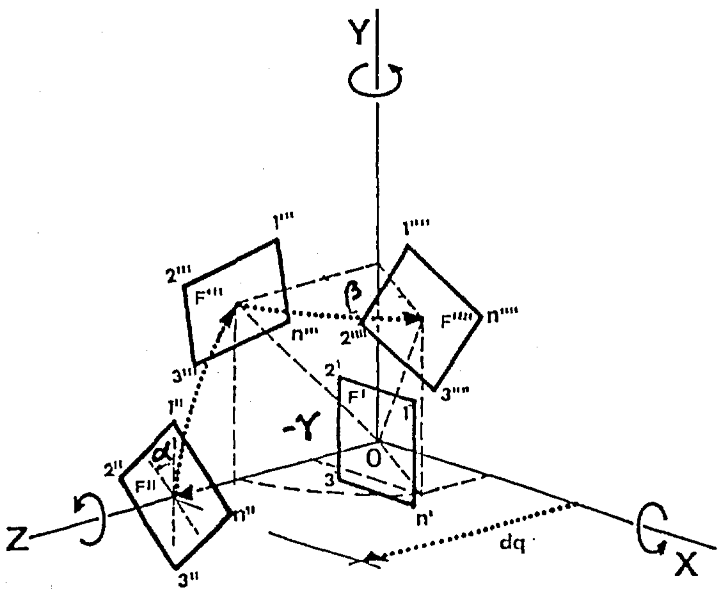


Figure 3.2: The various translation and rotation phases of a polygon in a polyhedron.

Thus for each individual polygon three successive rotations are carried out: around the *Z*-, *X*- and *Y*-axis, respectively in order to give it its final position. The distance of the translation of a specific polygon in a polyhedron is found from Figure 3.3.

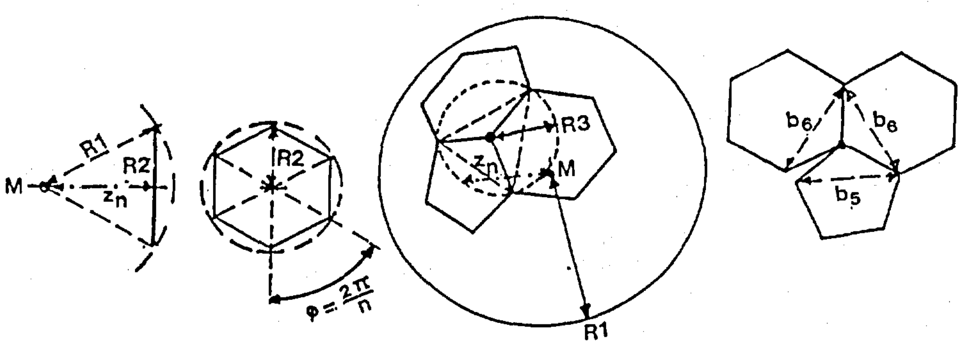


Figure 3.3: Vertex situation in a polyhedron, in this case formed of pentagons and hexagons.

Each n-gon has a circumscribed circle with radius $R2_n$ and lies at a distance Z_n from the system centre. The circumscribed sphere has the radius $R1$. $R3$ is the circumscribed circle of the vertex figure, which is in this case triangular as three polygons meet in this vertex.

$$\begin{aligned} R2_n &= 1/(2\sin(\pi/n)) \\ b_n &= 2\cos(\pi/n) \\ R1 &= 1/(2\sqrt{1-K^2}) \\ Z_n &= \sqrt{R1^2 - R2^2} \end{aligned}$$

The latter value is needed for the translation procedure that takes place over the distance dq in Figure 3.2.

10 different rotation cases were discerned, that each represent a specific combination of rotation cases and numbers of faces. With these combinations all known polyhedra can be formed not only but also their reciprocals, the star polyhedra and a number of other related figures. The rotations take place according:

1. the 4 triangles in the tetrahedron ($P1$)
2. the 6 squares in the cube ($P2$)
3. the 8 triangles in the octahedron ($P3$)
4. the 12 pentagons in the dodecahedron ($P4$)
5. the 20 triangles in the icosahedron ($P5$)
6. the 12 squares in the truncated cuboctahedron ($P11$)
7. the 30 squares in the truncated icosidodecahedron ($P17$)
8. (called 11) a regular n-gon
9. (called 12) the 2 n-gons in a prism ($P19$-23)
10. (called 13) the 2 n-gons in an antiprism ($P24$-28)

The programme requires for input:

$n1$ = the number of edges of the n-gon
$n3$ = the number of the rotation case
dk(3) = an eventual rotation around the Z-axis
dq(3) = an eventual translation along the Z-axis
Faktor = magnification ratio
KK(3) = eventual rotation around Z-, X- or Y-axis of the completed figure (in this order!)
QQ(3) = eventual translation along the X-, Y- or Z-axis of the completed figure

With one rotation case and one kind of polygons various solids can be described, just by varying the distance dq and the initial rotation dk (Fig. 3.4).

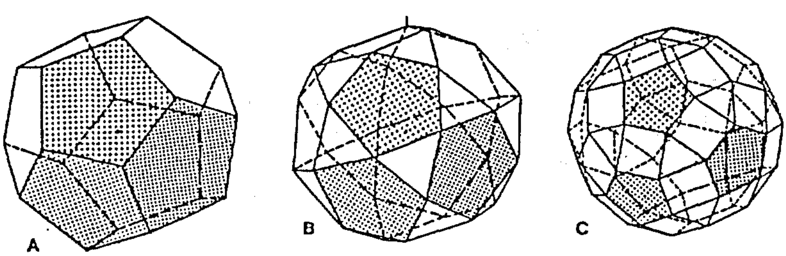


Figure 3.4: Formation of three different polyhedra with one rotation case and one kind of polygon.

For the regular polyhedra the rotation of one kind of polygon is sufficient, for other solids usually more than one run is necessary.

Figure 3.5 shows the formation of the squares and the decagons in a truncated icosidodecahedron.

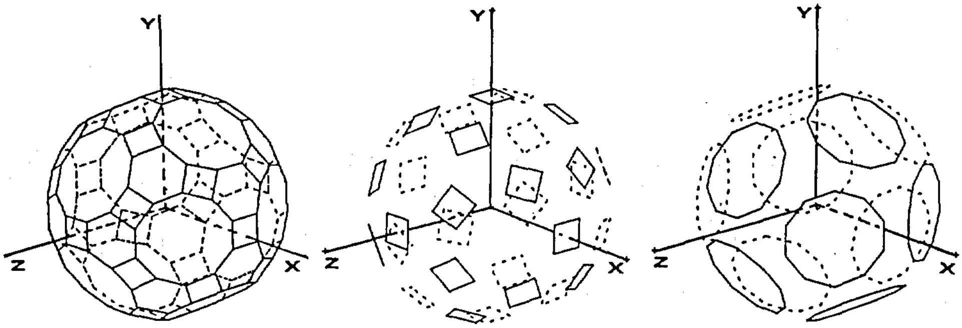


Figure 3.5: Rotation of two different polygons in *P*17.

In this way sketches of the uniform polyhedra could be made, at wish all in the same isometric projection (Figs. 3.6 and 3.7).

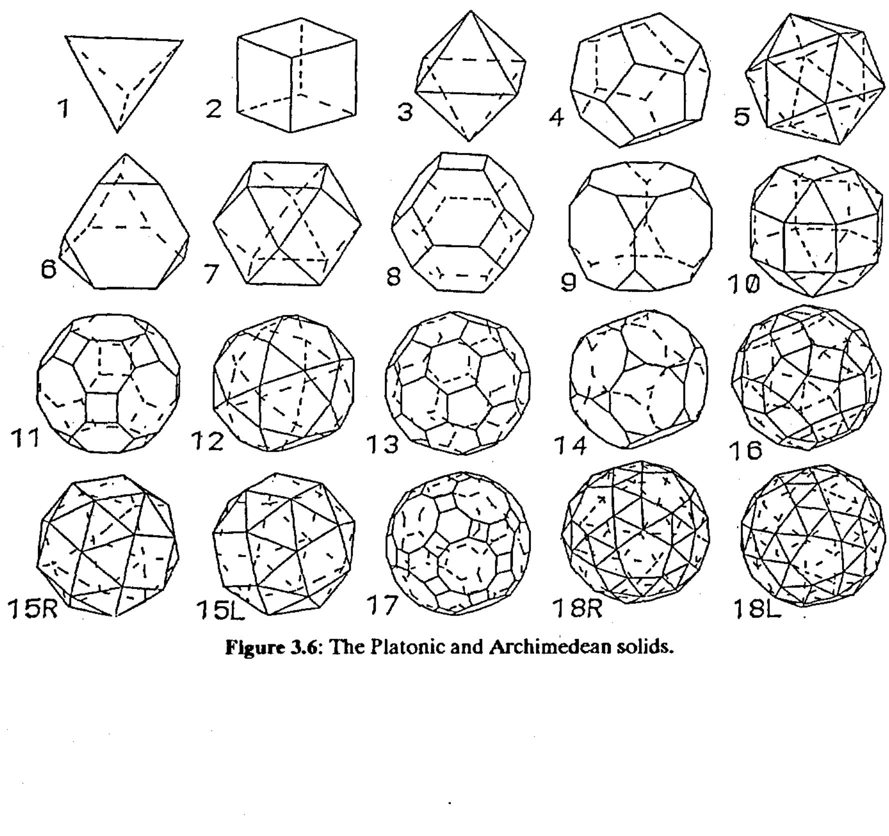


Figure 3.6: The Platonic and Archimedean solids.

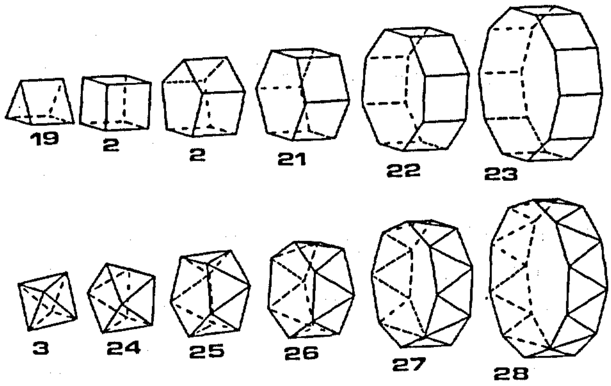


Figure 3.7: The uniform prisms and antiprisms.

The various options of the programme open the way to numerous interesting manipulations.

The calculation runs of one session are kept in memory so that intermediate relations – such as distances, face angles and dihedra angles – can be determined not only within one polyhedron but also between different polyhedra. On the other hand complete solids can be rotated and translated in space, for instance to form fitting patterns as in close-packings.

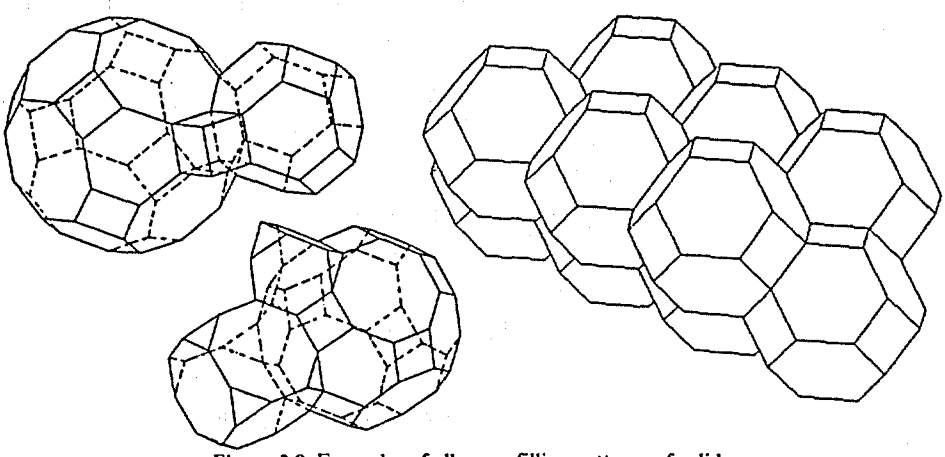


Figure 3.8: Examples of all space filling patterns of solids.

Choice of the input figure

The number of vertices of the polygons can be chosen between 1 and 50. For the description of the uniform polyhedra 3-, 4-, 5-, 6-, 8- and 10-sided polygons are sufficient, but increasing the number of sides gives a closer approximation to the circle (Fig. 3.9).

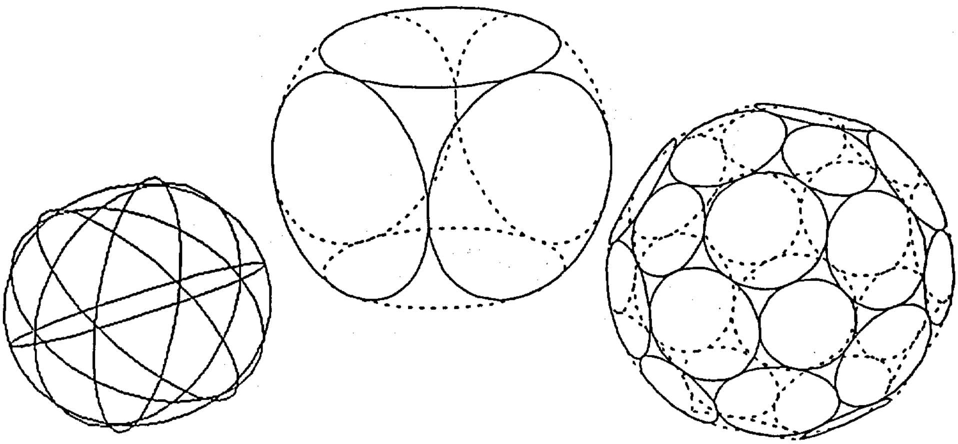


Figure 3.9: Substituting the polygons by touching circles.

The number of corners can be decreased also: a 2-gon is a line of unit length and a 1-gon is a point. If a point is rotated with a particular rotation case in fact the centre of the original polygon is rotated. If the distance is properly chosen, in this way the vertices of the dual or reciprocal figure are found. If such a figure – which is generally called a polar figure – is desired, the footpoints of the pyramidal caps that constitute it, have to be computed as well. Figure 3.10 represents a compound of two regular dual figures. It is composed of 20 triangular and 12 pentagonal pyramids. All dual solids can be generated this way.

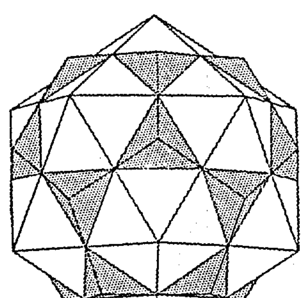


Figure 3.10: Compound of icosahedron and dodecahedron.

Figure 3.11 shows the results of this procedure. The star-shaped polyhedra can be formatted similarly, but a more direct method is available that produces better pictures.

Apart from the polygon input option two other options are available. The second option implies star-polygons or polygrams. This delivers a complete new set of figures (Fig. 3.13).

The third option gives the opportunity to formulate any desired spatial figure by putting in the coordinates of its vertices and defining its constituting faces. Such a figure can at wish be stored in a file and later on be used again. This expands the possibilities of the programme enormously. The stars of Figure 3.14 are pictured this way by successive storing and rotating of their constituting pyramidal caps. One of these is shown in an exploded form in order to show how it is generated.

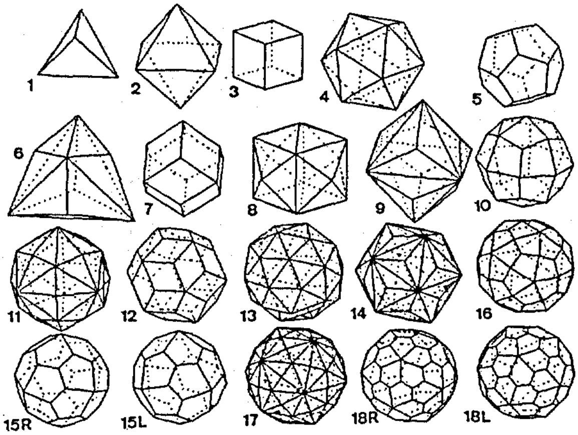


Figure 3.11: Reciprocal or dual solids of the Platonic and Archimedean solids, computed as polar figures.

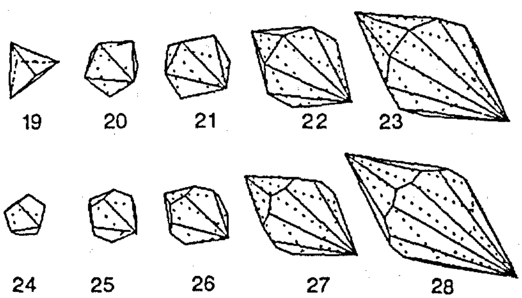


Figure 3.12: Reciprocals of prisms and antiprisms.

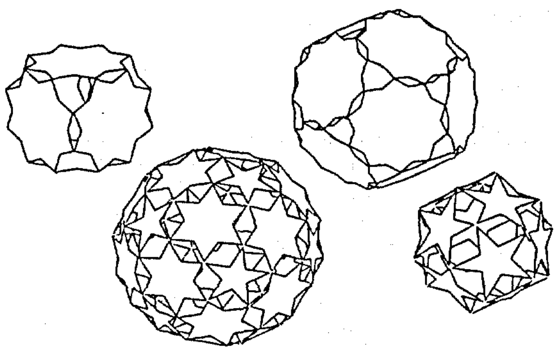


Figure 3.13: Solids with polygram faces.

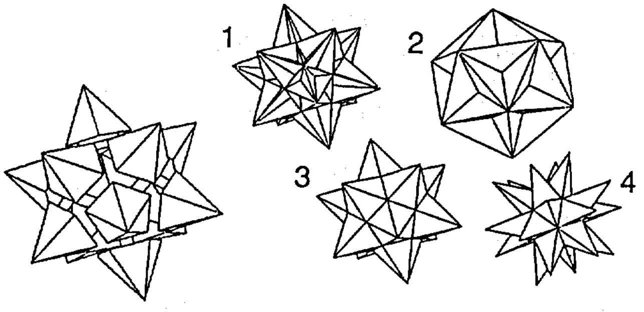


Figure 3.14: Stars formed by rotation of pyramidal caps.

Combinations of thus formed figures can be made in one picture with commonly found polyhedra. In Figure 3.15 small cubes or prisms are placed on the surface of polyhedra to show their relation with others.

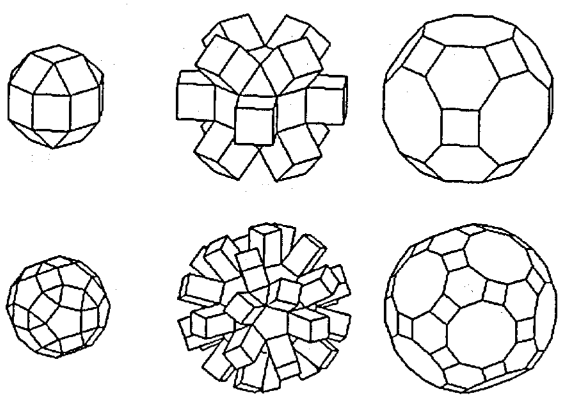


Figure 3.15: Cubes and prisms placed on $P10$ and $P16$ in order to form $P11$ and $P17$.

At the end of the procedure the decision can be made to store the complete figure in the previously mentioned so-called 'BIB'-file. This opens the opportunity to rotate complete figures so that systems of figures are formed. A few examples are shown in Figure 3.16. Such systems can be stored again and rotated again, a.s.o. An example will be shown at the end of this chapter in Figure 3.30.

Three-axial superellipsoids

In the uniform polyhedra all vertices are dispersed on the surface of a sphere. This purely centrally symmetric shape can be distorted by applying different magnification ratios in X-, Y- and Z-direction. This is one of the options in the previous polyhedron method, but this effect can be approached in a more general way according a suggestion by H. Kenner (1976).

A sphere can be considered as to be the result of a vertical circle that is rotated around a vertical axis (the Z-axis in Figure 3.17). As first step the radii of the figure can be chosen differently in X-, Y- and Z-direction. The circles take the form of 2 coherent ellipses.

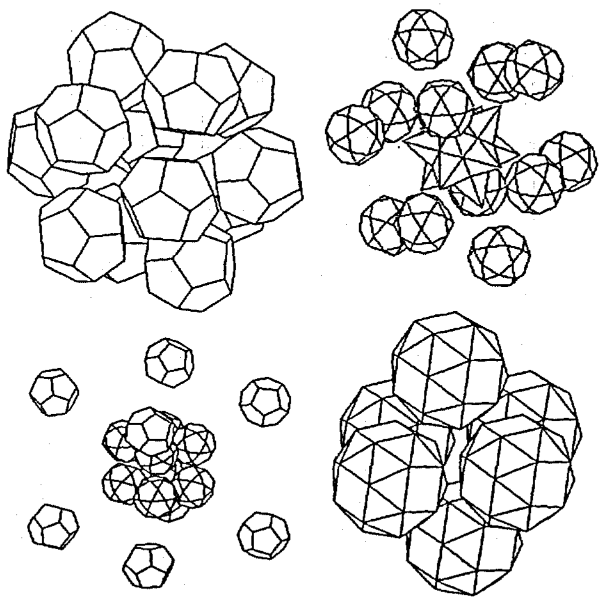


Figure 3.16: Examples of rotated polyhedra.

The ellipse is a second power equation. This power of 2 can be varied also. The effect of this is shown in Figure 3.18.

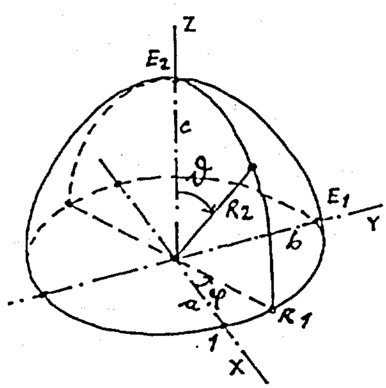


Figure 3.17: Scheme of 3-axial ellipsoid.

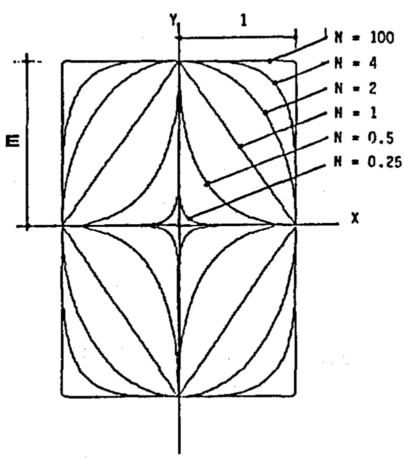


Figure 3.18: Ellipses of different powers.

The general equation of the ellipse is

$$(X^n / a^n) + (Y^n / b^n) = 1$$

If to n is given a higher value than 2, a closer approximation is found to the circumscribed rectangle. Such an ellipse is called a superellipse. For values lower than 1 the ellipse obtains a hollow – more or less star shaped – form. For the critical value 1 a rhombus is formed by straight lines that interconnect the maxima on the axes. If the ratio of the two axes a and b is said to be $E = a/b$ and if the equation of the radii in the horizontal and the vertical ellipse is found.

$$R1 = E_1/(E_1^{n1}\sin^{n1}\phi + \cos^{n1}\phi)^{1/n1}$$
$$R2 = R1 \cdot E_2/(E_2^{n2}\sin^{n2}\theta + R1^{n2}\cos^{n2}\theta)^{1/n2}$$

Varying n for the horizontal ellipse as well as for the vertical ellipse gives a great many different possible combinations. A matrix is given in Figure 3.19.

In this review the sphere is a special case of the figure CG (1 out of 16 principally different possibilities).

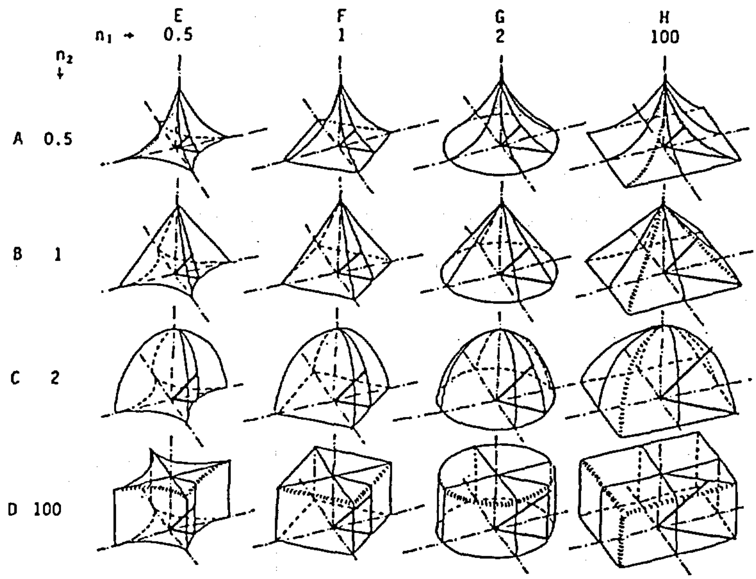


Figure 3.19: Matrix of different combination possibilities with varying power of horizontal and vertical ellipse.

It appears from this matrix that, apart from the sphere, many other interesting – and sometimes well known – shapes are formed, such as the pyramid, the cone, cylinder, cube, etc.

In order to visualize these shapes it is necessary to create a suitable subdivision pattern into its surface. A number of options is available, called classes:
1. Geodesic;
2. Similar to 1, but all vertices on horizontal circles;
3. Intersecting meridians and parallel circles;
4. Left- and right handed Schwedler subdivision;
5. Lamella system.

The geodesic method is based on one of the 3 Platonic solids with triangular faces: the tetrahedron, octahedron, or the icosahedron. At this moment the second only is used, the others are in preparation. Each one of the triangles can be projected upon the envelope after a proper subdivision in the desired frequency.

This can be done starting from two methods (Fig. 3.20).
1. Subdivision of the original triangular edge in equal parts and than 'exploding' the found vertices upon the 'sphere'.
2. Subdivision of the spherical angle in equal parts so that in the case of the sphere equal chords are found.

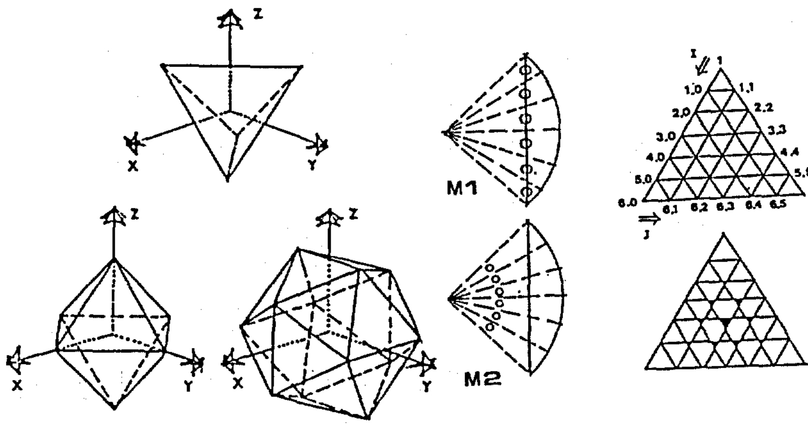


Figure 3.20: Methods for subdividing the spherical triangle.

These two methods can be used for the horizontal and vertical chords of classes 1, 2 and 3 and for the vertical chords of classes 4 and 5. The internal subdivision class 1 and 2 follows from Figure 3.20.

Once the subdivision of the original triangle edges is found in class 1 and 2 the internal vertices are determined by drawing straight lines in the original triangles between subsidiary points on opposite edges. In method 2 small triangular 'windows' are found (Borrego, 1968). Their centres are projected on the envelope.

The octahedron, if seen from above, has 4 sectors. The number of these sectors, called $n3$, can be changed, thus offering new possibilities.

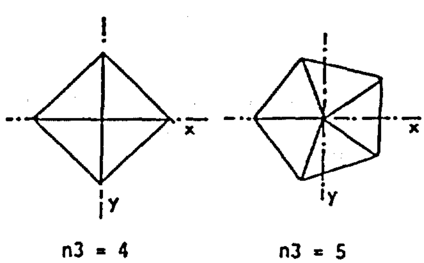
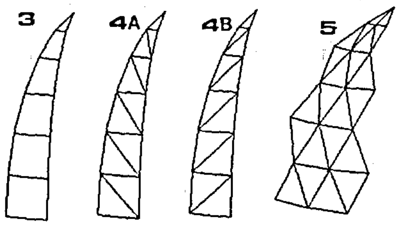


Figure 3.21: Varying the number of sectors.

This aspect is also used for the horizontal subdivision in classes 3, 4 and 5 (Fig. 3.22).

The points found in classes 1, 2, 3 and 5 can be interconnected in two ways, called figtypes 1 and 2 (Fig. 3.23).



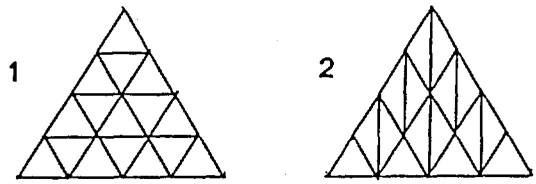


Figure 3.22: Reproduction of one sector in classes 3, 4 and 5.
Figure 3.23: Interconnection of vertices according Figtypes 1 and 2.

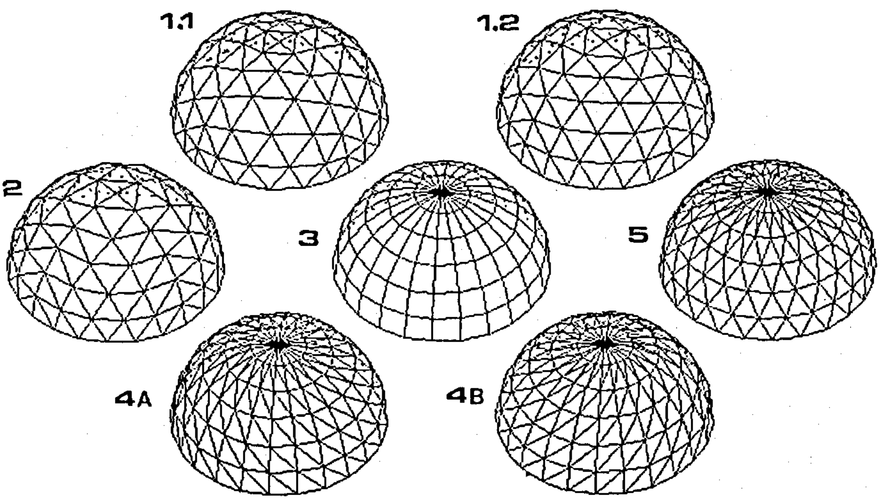


Figure 3.24: Different classes of subdivision in Figtype 1.

Adopting the octahedron as a starting point and using method 1 and Figtype 1 for the subdivision, the matrix of Figure 3.19 can be worked out in the way as shown in Figure 3.25.

Each figure in this matrix represents in fact one possibility out of a group of different shapes, as still variations are possible in:

- frequency of subdivision
- class
- method
- number of sectors
- ratio of axes

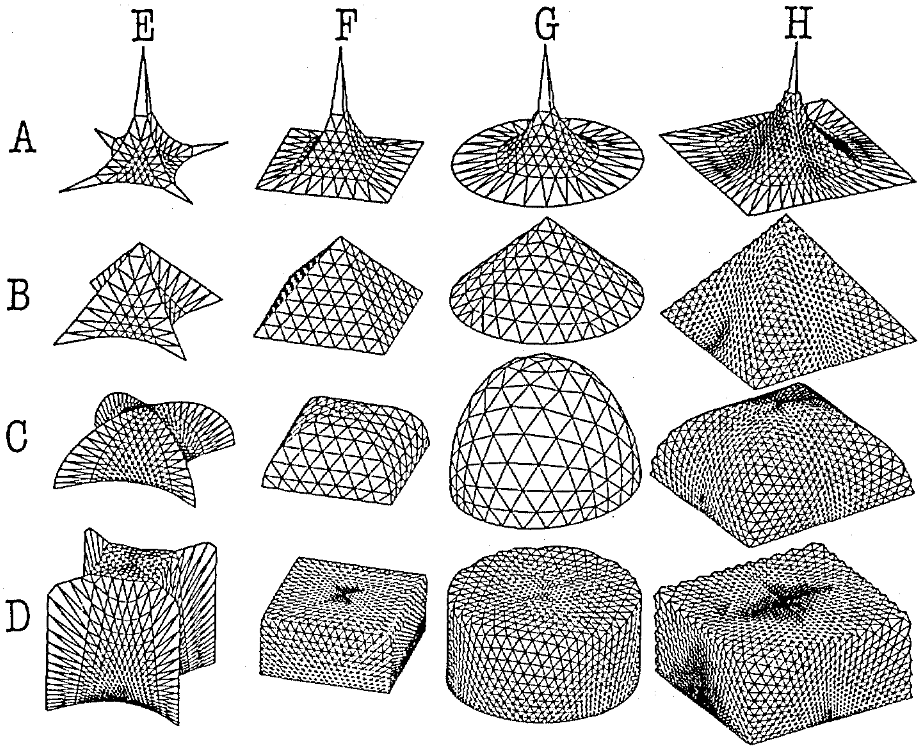


Figure 3.25: The matrix of Fig. 3.19, worked out in geodesic patterns.

The effect of the Figtype 2 pattern is demonstrated in Figure 3.26.

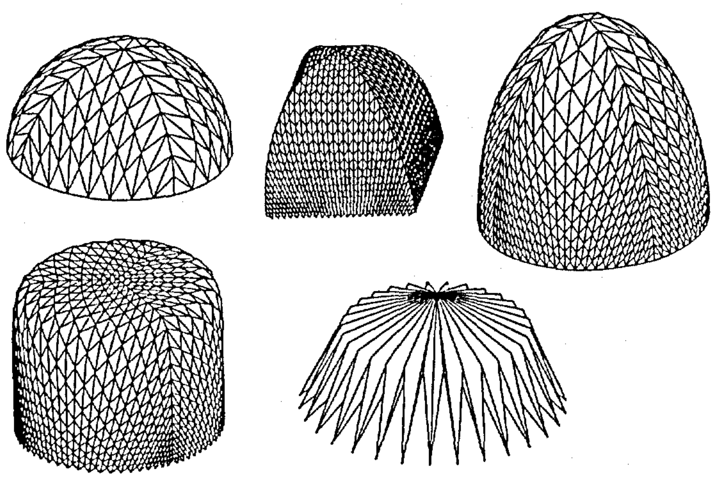


Figure 3.26: The effect of Figtype 2-connections.

This Figtype 2 way of connecting reminds of the antiprismatic vaults, shown in Question 2. For a low frequency and a great number of sectors interesting folded shapes are found. Variation in $E1$ and $E2$ can lead to such extremities as in Figure 3.27.

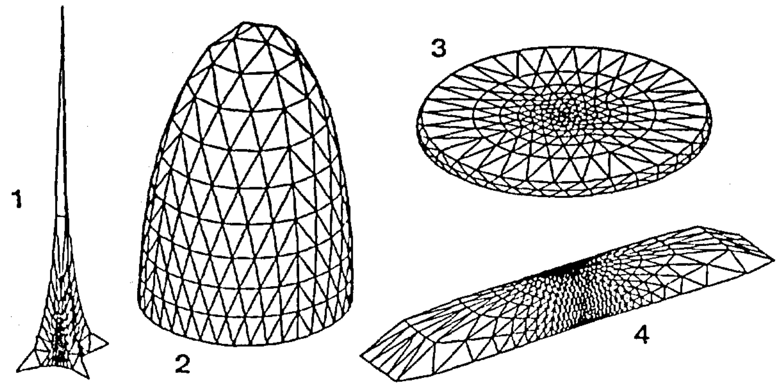


Figure 3.27: Variation in ratios of axes a, b and c.

Some of the shapes generated in this way remind of crystalline elements or fractions in aggregated materials, such as f.i. concrete.

This method may therefore create openings to the representation of such materials (Stroeven, Huybers, and Arends, 1983).

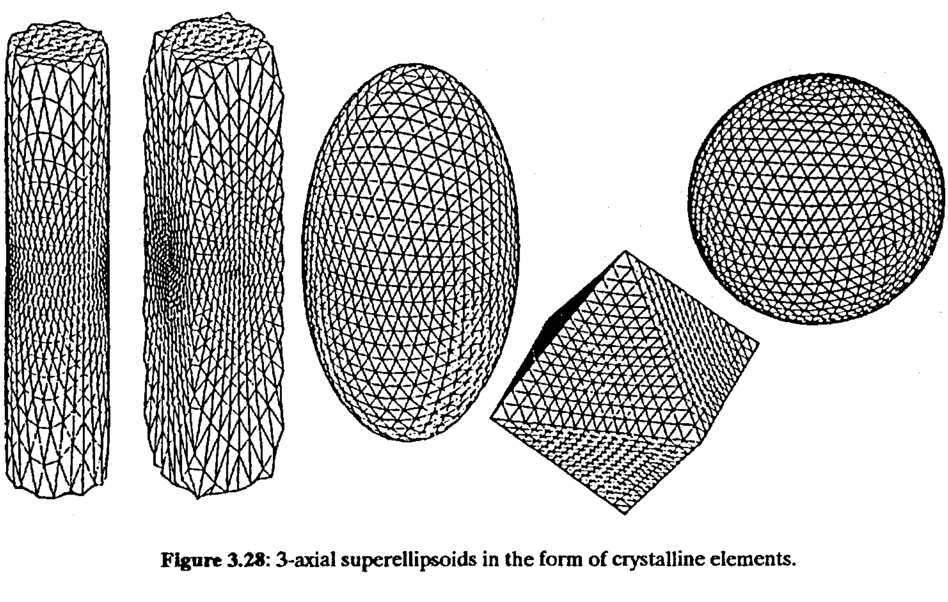


Figure 3.28: 3-axial superellipsoids in the form of crystalline elements.

Alphanumeric presentation

The data obtained from the previous procedures can at wish be presented in alphanumeric form. For the polyhedra the coordinates of the vertices are produced and other data like distances, face angles and dihedral angles are found interactively. In the case of the superellipsoids, all relevant data are printed upon order together with a number of statistical values.

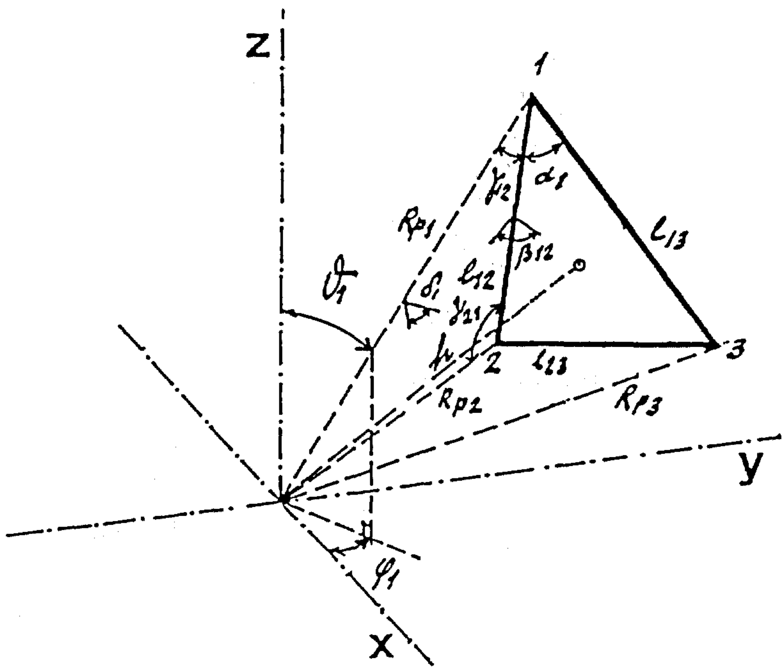


Figure 3.29: Relevant data of triangular element that are produced in alphanumeric form.

Further extensions

The ellipsoid part of the programme calculates the magnitude of volumes and of surface areas, the latter in three projections. This opens the opportunity to enhance material properties and loadings to these areas, so that input data become available for calculations with respect to strength, stiffness and other physical aspects. If the superellipsoid path is followed, the shape that is found can again be stored in the earlier mentioned 'BIB'-file. This file contains the spatial elements that can be used as the starting point of a rotation procedure, so that it can be started over again, this time with the ellipsoid of the basic element.

It is then projected in space as many times as corresponds with the rotation case chosen. This result can be stored again, rotated, stored a.s.o. This may lead to huge complex, spatial configurations with a magnitude depending on the available computer facilities. In this case a MEGA-Atari ST4 was used with a 40 Mb Ramdisc, but it also works – maybe slightly slower – on an Atari-1040 ST. Figures 3.30 and 3.31 show a few examples of this 'mirroring' in space.

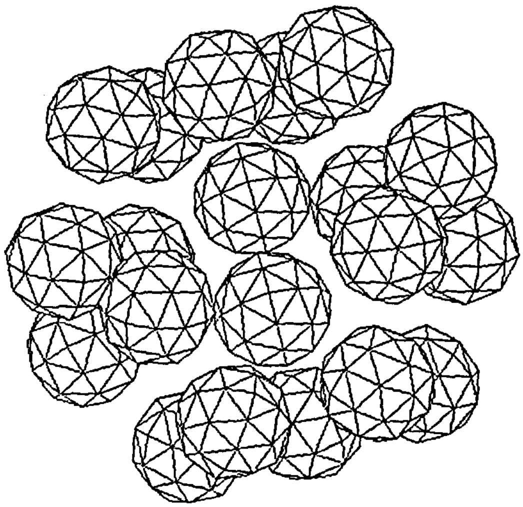


Figure 3.30: Number of spheres, rotated in space.

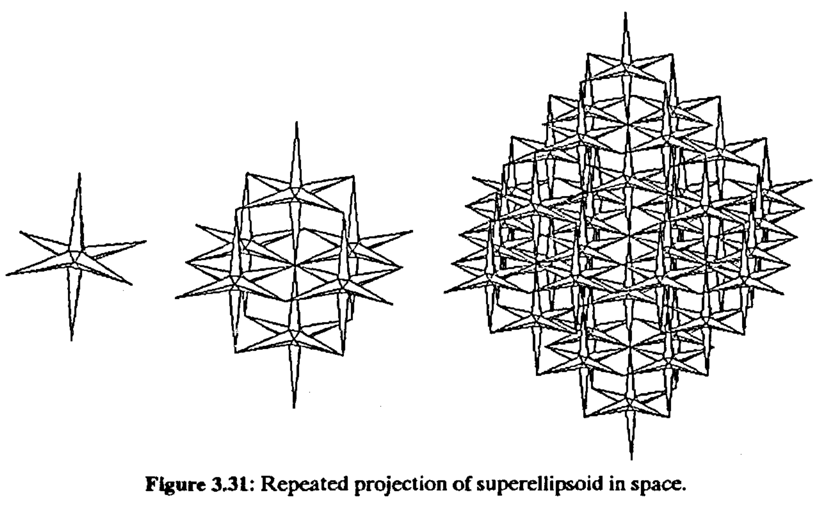


Figure 3.31: Repeated projection of superellipsoid in space.

Presentation

The figures can be obtained either in parallel or in perspective projection, as a wire model or in hidden line, on a monitor in black or white or eventually in color and hardcopied as a screen-dump or as a line-plot. The screen-dumps have gained very much in quality with the availability of laser-printers. Even color prints are within reach – though relatively expensive at the moment.

If 2 perspective sketches of the same object are made at distances according to the eye-distance, 3-dimensional pictures can be made in the form of anaglyphs or other stereoscopic techniques. The two pictures shown in Figure 3.32 can be seen this way.

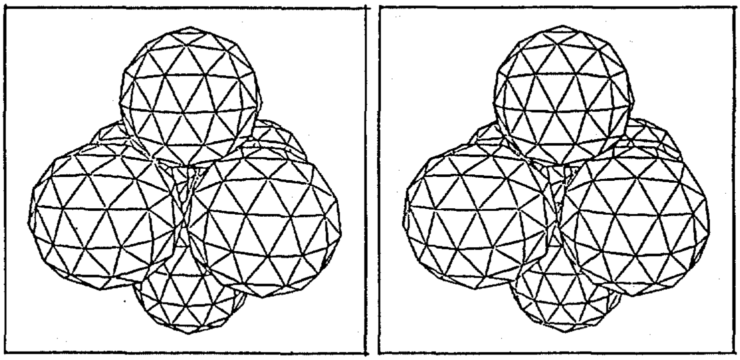


Figure 3.32: Stereoscopic picture of sphere-arrangement.

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