OUTSIDE IN

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The video *Outside In* illustrates an amazing mathematical discovery made in 1957 by Steve Smale: you can turn the surface of a sphere inside out without making a hole, if you think of the surface as being made of an elastic material that can pass through itself. Communicating how this process of *eversion* can be carried out has been a challenge to differential topologists ever since.

Outside In uses nontechnical language and computer animation to illustrate the process and to explain the concepts involved to a nonmathematical audience. Yet the video retains mathematical depth: we introduce the concept of a "regular homotopy" from topology, which is traditionally not encountered until advanced undergraduate mathematics classes. (Topology is the study of the properties of objects that remain unchanged even if the surface can be deformed, and differential topology is the study of smooth objects without corners or bends.) The metaphor we use is that of a material that can stretch and pass through itself, but that self-destructs if punctured or even pinched sharply. Of course, there is no such material in real life! That's where computer graphics comes in.

Why Turn a Sphere Inside Out?

The short answer is that it is a mathematical puzzle that is interesting and counter-intuitive, and therefore challenging to solve. It is an abstract problem, unrelated to objects in the natural world.

But it also stems from a classification problem of surfaces and deformations, more specifically of "immersions", or smooth surfaces without kinks. In these immersions, the notion of "sidedness" is also encoded into the surfaces. This property of sidedness is preserved no matter which side of the sphere is inside or out. A by-product of this was Smale's idea that it should be possible to turn a sphere inside out without poking a hole in it. People ask--"Well, why *can't* you make a hole in it?" The answer is that one of the points is to find equivalence between surfaces, to find out which objects are topologically the same, so the hole invalidates the process.

History

For a while, Smale's discovery was met with skepticism. The famous mathematican Raoul Bott, who had been his graduate adviser, flatly told Smale that he was wrong, and explained why he thought so. Later he became persuaded that Smale's reasoning was correct, but he---like many other mathematicians---was still frustrated by the abstractness of Smale's proof, and wished to see a more direct, practical, step-by-step, sphere eversion. After this acceptance, the race was on to find a method which could show the actual eversion process.

It took seven years before Arnold Shapiro found a practical way to do it. Since the problem remained hard to visualize, more methods were invented later, by Bernard Morin and several others. The May 1966 Scientific American cover article by Anthony Phillips first brought sphere eversion to the attention of the non-mathematical public. However, even with a series of drawn images, the problem remained unclear. The problem with the Scientific American eversion was not that it was not that the explanation or the figures were poor, for they weren't. It is rather that a complete mental picture of the evolving sphere is very hard to keep in mind, and clear printed images are even harder to draw, because of the many layers of surface. The most successful expositions of the everting sphere concentrate on local details, and rely on the viewer's powers of synthesis to piece the details together.

Bill Thurston developed his idea of using corrugations, which you will see in the video. This method, invented in 1974, is actually a general approach that can often be applied to other surfaces in any dimension. It has also shown to be the most easily understood visually.

In the computer graphics realm, Nelson Max was the pioneer of mathematical visualization movies with his ground-breaking Topology Films project. Computer graphics were first used to make a mathematical movie despite the primitive state of hardware and software. His film *Turning a Sphere Inside Out* illustrates Morin's method of eversion.

Structure and Symmetry

The framework of the video is a dialogue between a female teacher and a male student. In the first scene they work out between themselves the ground rules of what it means to turn a sphere inside out. The first is that the material is an abstract mathematical material, not a real-world object like a ball or a balloon. This material can stretch and bend, and pass through itself, but cannot be ripped, punctured, or creased. That means that simply pushing the caps straight through one another won't work either--a tight loop or crease will form around the equator. How can this work? The student remains skeptical that the problem can be solved under these rules. If anything his skepticism increases in subsequent scenes, as the teacher persuades him that a circle *cannot* be turned inside out under the same rules. However, an idea that is introduced in connection with curves---namely, adding waves, or corrugations---turns out to be useful for surfaces as well.

To further understand the process, we (and our student) are shown that loops can turn into twists. (This can be demonstarted in the "real world" with a belt.) And, since our abstract material is as stretchy as we want it to be, we can then add the aforementioned corrugations to the surface. These symmetrical corrugations absorb the creases that will form around the equator when the caps are pushed through one another. Although the process looks quite complex at first glance, the motion can be broken down into stages. First, we corrugate the sphere. Next, we push the poles through each other, stopping when we create a loop at the equator. Now, we twist the two caps in opposite directions to convert the loops in the middle to twisting at the caps. We then push the equatorial region of the surface radially through the center. Finally, we uncorrugate to complete the eversion.

Since the construction is symmetrical, we can illustrate it with just a portion of the surface. It turns out that in our construction, the sphere can be broken into 16 copies of a single piece, which extends from a pole to the equator. (Imagine a peeled orange with 8 sections arranged longitudinally. Then slice through the orange at its "equator".) This sixteenth is the fundamental building block of the eversion—the entire sphere can be created by rotating copies of this piece.

Symmetries are apparent all through the video—in the eversion process itself, in the construction of the sphere, in the simple equations we show, and in all the patterns that are produced during the process. The division of the sphere into equatorial strips during subsequent eversions reveal another level of symmetries within which hitherto were hidden.

One of our main goals in making this video is to demonstrate that seeing rather than reading about it produces comprehension. It is at this point where words are inadequate and the video must be shown!

In Conclusion

Outside In is the result of collaboration between mathematicians, programmers, and designers. We developed a great deal of custom software in addition to using RenderMan, Softimage, and Mathematica. I want to thank Silvio Levy and Tamara Munzner (the other two directors), Nathaniel Thurston, Stuart Levy, David Ben–Zvi, Daeron Meyer, and all of the other contributors. Thanks to Silvio Levy and Tamara Munzner in the writing of this brief description.

"The Geometry Center" is the informal name for the National Science and Technology Center for the Computation and Visualization of Geometric Structures, based at the University of Minnesota.

The author is an artist and designer who served as one of three directors on this video. She also served as art director on the previous Geometry Center video "Not Knot".