

ON MAPPING THE EQUILIBRIUM EQUATIONS OF A CLASS OF THIN
SHALLOW SHELLS WITH VARYING CURVATURE TO CONSTANT
COEFFICIENT EQUATIONS VIA LIE GROUPS

VASSILEV Vassil Mitev

Laboratory of Hydroelasticity,
Institute of Mechanics & Biomechanics,
Bulgarian Academy of Sciences,
Acad. G. Bonchev Str., Bl. 4, 1113 Sofia, BULGARIA.

INTRODUCTION

The symmetry of a structure is in essence a form of invariance of this structure under some type of transformations. The acceptance of such an idea and its consistent application seems to be the only way a sense or a feeling for the existence of symmetry to be developed into a well-build theory.

The investigation on symmetry associated with partial differential equations (p.d.eqns) in the above sense is pioneered by Sophus Lie. Recently his theory has been significantly advanced notably by L.V. Ovsiannikov and it is known now as group analysis of differential equations (see Ref. [1]). In principle this theory is based on the concept of invariance of p.d.eqns under continuous (Lie) groups of transformations.

In this paper an application of the above mentioned theory is presented. The equilibrium eqns of one class of thin elastic shells with varying curvature are considered and it is shown how the symmetry of this eqns can be utilized for their solving.

FUNDAMENTAL EQUATIONS

Let the surface F in the three-dimensional Euclidean space $\mathbb{R}^3$ be given by the eqn

$$(1) \quad z = f(x, y) = (x^2 + y^2)^{-2} [A(x^2 - y^2) + 2Bxy],$$

where A and B are real constants and (x, y, z) is a Cartesian coordinate system in space $\mathbb{R}^3$. The coordinate variables x, y will be used further as Gaussian coordinates in F .

Let us consider now a thin elastic shell with constant cylindrical rigidity D and thickness h and let its middle surface coincide with some part of the surface F .

Since the surface F is evidently asymptotically flat,

i.e. $f \rightarrow 0$ when $x^2 + y^2 \rightarrow 0$, then for an appropriate number R the following estimates are valid

$$(\partial f / \partial x)^2 \ll 1, \quad (\partial f / \partial x)(\partial f / \partial y) \ll 1, \quad (\partial f / \partial y)^2 \ll 1$$

in the domain $\Omega = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq R\}$.

Hence, in terms of Donnell-Vlasov theory (see e.g. Ref. [2]) the shell under consideration can be treated as shallow in the domain Ω . Then, in linear treatment, the system of its equilibrium eqns can be written in the following form

$$(2) \quad \begin{aligned} D\Delta\Delta w - b_{11}\partial^2\phi/\partial y^2 + 2b_{12}\partial^2\phi/\partial x\partial y - b_{22}\partial^2\phi/\partial x^2 &= q(x, y), \\ (1/Eh)\Delta\Delta\phi + b_{11}\partial^2 w/\partial y^2 - 2b_{12}\partial^2 w/\partial x\partial y + b_{22}\partial^2 w/\partial x^2 &= 0, \end{aligned}$$

where w is the transversal displacement function, ϕ is the Airy's stress function, E is the Young modulus, q is the external transversal loading, $\Delta \equiv \partial^2/\partial x^2 + \partial^2/\partial y^2$ and $b_{\alpha\beta}$ ($\alpha, \beta = 1, 2$),

$$(3) \quad b_{11} = \partial^2 f / \partial x^2, \quad b_{12} = b_{21} = \partial^2 f / \partial x \partial y, \quad b_{22} = \partial^2 f / \partial y^2$$

are the components of the curvature tensor of F , approximated in the manner accepted in the shallow shell theory. Accordingly to formulae (1) and (3), the explicit form of components $b_{\alpha\beta}$ is

$$(4) \quad \begin{aligned} b_{11} &= -b_{22} = (x^2 + y^2)^{-4} [A(x^4 + y^4 - 6x^2y^2) + 4Bxy(x^2 - y^2)], \\ b_{12} &= b_{21} = (x^2 + y^2)^{-4} [4Axy(x^2 - y^2) - B(x^4 + y^4 - 6x^2y^2)]. \end{aligned}$$

The complicated form (4) of the coefficients $b_{\alpha\beta}$ makes the integration of system (2) quite problematic. A considerable progress in this direction could be achieved however by using the symmetry of this system of eqns.

FORMULATION OF THE PROBLEM

When a system of p.d.eqns with varying coefficients is being solved it is very helpful to be known whether there is a transformation of its independent and dependent variables so that after rewriting the system in the new variables its coefficients to become constant. Indeed, if the system could be transformed in this manner, then each one of the well known methods for integration of systems with constant coefficients may be used for its solving.

In Ref. [4] Bluman shows that when the full symmetry group of a given linear homogeneous p.d.eqn with varying coefficients is known then a definitive and constructive answer of the above question can be found. Following his approach in general outline we analyse this problem below for the system (2).

Obviously, the answer of the problem in question does not depend on the form of the right hand side of system (2). In view of that it is naturally the simplest system of this form, i.e. the homogeneous one, to be considered.

SYMMETRY GROUP

Let us denote by S the homogeneous system of eqns of form (2). In Ref.[3] it is shown that the full symmetry group of system S is two-parameter Lie group of transformations in the space $\mathbb{R}^4(x, y, w, \phi)$ of the independent x, y and the dependent w, ϕ variables of system S . The infinitesimal generators of this group are

$$\begin{aligned} X_1 &= (x^2 - y^2) \partial / \partial x + 2xy \partial / \partial y + 2xw \partial / \partial w + 2x\phi \partial / \partial \phi, \\ X_2 &= -2xy \partial / \partial x + (x^2 - y^2) \partial / \partial y - 2yw \partial / \partial w - 2y\phi \partial / \partial \phi. \end{aligned} \quad (5)$$

TRANSFORMATION OF THE FUNDAMENTAL EQUATIONS

Consider the following transformations of the variables of system (2):

$$\begin{aligned} x' &= f_1(x, y), \quad w' = W_1(x, y)w + F_1(x, y)\phi, \\ y' &= f_2(x, y), \quad \phi' = W_2(x, y)w + F_2(x, y)\phi, \end{aligned} \quad (6)$$

where f_α, W_α and F_α ($\alpha = 1, 2$) are arbitrary, but sufficiently smooth functions in the domain Ω . Evidently only the transformations of form (6) provide linearity and homogeneity of system S after passing to the new variables. That is the reason only transformations of this special type to be considered.

If there exists a transformation of form (6) such that the infinitesimal generators of the symmetry group of system S take on in the new variables the form

$$X'_1 = \partial / \partial x', \quad X'_2 = \partial / \partial y', \quad (7)$$

then after rewriting system S in this new variables its coefficients become constant and vice-versa. This conclusion is a consequence from the following well known general group property of the linear homogeneous systems of p.d. eqns: if a linear homogeneous system of p.d. eqns has constant coefficients then this system admits the group of translations in the space of its independent variables and vice-versa.

On the other hand in passing to the new variables the infinitesimal generators of the symmetry group of system S transform (see e.g. Ref.[5], p.45) as follows

$$\begin{aligned} X'_\alpha &= (X_\alpha x') \partial / \partial x' + (X_\alpha y') \partial / \partial y' + \\ &+ (X_\alpha w') \partial / \partial w' + (X_\alpha \phi') \partial / \partial \phi' \quad (\alpha = 1, 2), \end{aligned} \quad (8)$$

Comparing the right hand sides of (7) and (8) and sub-

stituting (5) and (6) in the obtained expressions we work out the following overdetermined linear system of first-order p.d. eqns

$$(9) \quad \begin{aligned} (x^2-y^2)\partial f_\alpha/\partial x + 2xy\partial f_\alpha/\partial y &= \delta_{\alpha 1}, \\ 2xy\partial f_\alpha/\partial x - (x^2-y^2)\partial f_\alpha/\partial y &= -\delta_{\alpha 2}, \\ (x^2-y^2)\partial W_\alpha/\partial x + 2xy\partial W_\alpha/\partial y + 2xW_\alpha &= 0, \\ (x^2-y^2)\partial F_\alpha/\partial x + 2xy\partial F_\alpha/\partial y + 2xF_\alpha &= 0, \\ 2xy\partial W_\alpha/\partial x - (x^2-y^2)\partial W_\alpha/\partial y + 2yW_\alpha &= 0, \\ 2xy\partial F_\alpha/\partial x - (x^2-y^2)\partial F_\alpha/\partial y + 2yF_\alpha &= 0, \end{aligned}$$

where $\alpha, \beta = 1, 2$ and $\delta_{\alpha\beta}$ is the Kronecker delta symbol.

Combining all the above we are able to formulate now the following result: the system of form (2) can be transformed to constant coefficient system by transformation of form (6) if and only if the corresponding functions f_α , W_α and F_α satisfy system (9).

It is easy to show that the functions

$$\begin{aligned} f_1 &= -x(x^2+y^2)^{-1}, \quad f_2 = y(x^2+y^2)^{-1}, \\ W_1 &= F_2 = (1/4)(x^2+y^2)^{-1}, \quad W_2 = F_1 = 0, \end{aligned}$$

satisfy system (9). The corresponding transformation of form (6) is

$$(10) \quad \begin{aligned} x' &= -x(x^2+y^2)^{-1}, \quad w' = w(1/4)(x^2+y^2)^{-1}, \\ y' &= y(x^2+y^2)^{-1}, \quad \phi' = \phi(1/4)(x^2+y^2)^{-1}. \end{aligned}$$

Rewriting system (2) in the variables given by (10) we get

$$\begin{aligned} DA'\Delta'w' - A\partial^2\phi'/\partial y'^2 - 2B\partial^2\phi'/\partial x'\partial y' + A\partial^2\phi'/\partial x'^2 &= q(x'^2+y'^2)^{-3}, \\ (1/Eh)\Delta'\Delta'\phi' + A\partial^2w'/\partial y'^2 + 2B\partial^2w'/\partial x'\partial y' - A\partial^2w'/\partial x'^2 &= 0, \end{aligned}$$

where $\Delta' \equiv \partial^2/\partial x'^2 + \partial^2/\partial y'^2$.

REFERENCES

1. Ovsianikov L.V. Group Analysis of Differential Equations. Moscow, Nauka, 1978 (in Russian), Translated in English by W.F.Ames - Academic, 1982.
2. Volmir A.S. Nonlinear Dynamics of Plates and Shells. Moscow, Nauka, 1972 (in Russian).
3. Vassilev V.M. Group Properties of a Class of Equations of the Thin Elastic Shell Theory. Annals of the University of Sofia, 1987, vol. 81, Part 2 - Mechanics (in Bulgarian).
4. Bluman G.W. On Mapping Linear Partial Differential Equation to Constant Coefficient Equation. SIAM J. on Appl. Math. (1983), December, Vol.43, No. 6, p. 1259-1273.
5. Ibragimov N.H. Groups of Transformations in Mathematical Physics. Moscow, Nauka, 1983 (in Russian).