

COLORED ANTISYMMETRY

Slavik V. Jablan
The Mathematical Institute
11001 Belgrade, Knez Mihailova 35, pp. 367
Yugoslavia

The concept of P -symmetry (permutation symmetry) introduced by A.M. Zamorzaev is defined as follows. If P is a subgroup of the symmetric permutation group of p indices, and G is a discrete symmetry group, every transformation $C=cS=Sc$, $c \in P$, $S \in G$ is a P -symmetry transformation. Every group G^P derived from G by such a substitution of symmetries by P -symmetries is a P -symmetry group. If the substitutions included in G^P exhaust the group P , G^P is a complete P -symmetry group. Every complete P -symmetry group G^P can be derived from its generating group G by means of searching in G and P for normal subgroups H and Q for which the isomorphism $G/H \cong P/Q$ holds, by paired multiplication of the cosets corresponding in this isomorphism and by the unification of the products obtained. The groups of complete P -symmetry fall into senior ($G=H$ and $G^P=G \times P$), junior ($G/H \cong P$ and $G^P \cong G$) and middle groups for $Q=P$, $Q=I$ and $I \subset Q \subset P$, respectively.

In the case of Belov (p)-symmetry, the group $P=C_p$ is generated by the permutation $c_1=(12\dots p)$ satisfying the relations:

$$c_1^p = I \quad c_1 S = S c_1, \quad S \in G.$$

In the case of Pawley (p')-symmetry, the group $P=D_{p(2p)}$ is the regular dihedral permutation group generated by the permutations c_1 and $e_1=(11')$ satisfying the relations:

$$c_1^p = e_1^2 = (c_1 e_1)^2 = I \quad c_1 S = S c_1 \quad e_1 S = S e_1, \quad S \in G.$$

In the case of the ($p2$)-symmetry, the group $P=D_p$ is the irregular dihedral permutation group generated by the permutations c_1 and $e_1=(12)$ satisfying the relations:

$$c_1^p = e_1^2 = (c_1 e_1)^2 = I \quad c_1 S = S c_1 \quad e_1 S = S e_1, \quad S \in G.$$

By the mathematicians of Kishinev school (Zamorzaev, Palistrant, Galyarskii, Karpova...) the crystallographic space P -symmetry groups G^P are derived for $P=C_p, D_{p(2p)}$ and D_p ($p=3,4,6$). After the small corrections made by the author for (62)-symmetry groups, their numbers are: 817 G^P ($111 G^3 + 327 G^4 + 379 G^6$), 3610 G^{P2} ($309 G^{32} + 1667 G^{42} + 1634 G^{62}$) and 2212 $G^{P'}$ ($309 G^{3'} + 950 G^{4'} + 953 G^{6'}$) ($p=3,4,6$).

If $P=C_2^l$ we have the simple ($l=1$) and multiple ($l \geq 2$) antisymmetry groups, where $P=C_2^l$ is generated by the antiidentities $e_1, e_2, \dots, e_l$ of the first, second, ..., l th kind satisfying the relations:

$$e_i^2 = I, \quad e_i e_j = e_j e_i, \quad (i, j=1, 2, \dots, 7)$$

and commuting with the symmetries of the generating symmetry group G .

The derivation of 7-multiple antisymmetry groups is realized by Zamorzaev & Palistrant, and completed by the author. There are $N_1=1191$, $N_2=9511$ [3], $N_3=109139$, $N_4=1640955$, $N_5=28331520$ [1] and 419973120 [3] junior 7-multiple antisymmetry groups of the M^p -type $\mathcal{G}^7$.

The next problem: derivation of crystallographic colored 7-multiple antisymmetry groups $\mathcal{G}^{1,p}$, $\mathcal{G}^{1,p'}$ and $\mathcal{G}^{1,p^2}$ ($p=3, 4, 6$) is solved by the use of generalized antisymmetric characteristic method (AC-method):

Definition 1. Let all the products of P -symmetry generators of a group $G^{(P)}$, within which every generator participates once at the most, be formed, and then the subsets of transformations equivalent with regard to P -symmetry, be separated. The resulting system is called the antisymmetric characteristic of the group $G^{(P)}$.

As the final result, for the complete crystallographic $(P, 7)$ -symmetry junior three-dimensional space groups of the M^p -type ($P=C_p$, $D_p(2p)$, D_p) the numbers $N_{\mathcal{G}}^P$ ($p=3, 4, 6$) are the following:

$$\begin{aligned} N_1^P &= 379 \mathcal{G}^{1,3} + 1705 \mathcal{G}^{1,4} + 2050 \mathcal{G}^{1,6} = 4134 \mathcal{G}^{1,P} \\ N_2^P &= 2050 \mathcal{G}^{2,3} + 11447 \mathcal{G}^{2,4} + 16234 \mathcal{G}^{2,6} = 29731 \mathcal{G}^{2,P}; \\ N_3^P &= 16234 \mathcal{G}^{3,3} + 90160 \mathcal{G}^{3,4} + 153720 \mathcal{G}^{3,6} = 260114 \mathcal{G}^{3,P}; \\ N_4^P &= 153720 \mathcal{G}^{4,3} + 645120 \mathcal{G}^{4,4} + 1249920 \mathcal{G}^{4,6} = 2048760 \mathcal{G}^{4,P}; \\ N_5^P &= 1249920 \mathcal{G}^{5,3} = 1249920 \mathcal{G}^{5,P}; \end{aligned}$$

$$\begin{aligned} N_1^{P'} &= 1634 \mathcal{G}^{1,3'} + 6361 \mathcal{G}^{1,4'} + 7288 \mathcal{G}^{1,6'} = 15283 \mathcal{G}^{1,P'}; \\ N_2^{P'} &= 13391 \mathcal{G}^{2,3'} + 53664 \mathcal{G}^{2,4'} + 78825 \mathcal{G}^{2,6'} = 145880 \mathcal{G}^{2,P'}; \\ N_3^{P'} &= 150197 \mathcal{G}^{3,3'} + 441924 \mathcal{G}^{3,4'} + 967568 \mathcal{G}^{3,6'} = 1559689 \mathcal{G}^{3,P'}; \\ N_4^{P'} &= 1888320 \mathcal{G}^{4,3'} + 2056320 \mathcal{G}^{4,4'} + 10321920 \mathcal{G}^{4,6'} = 14266560 \mathcal{G}^{4,P'}; \\ N_5^{P'} &= 19998720 \mathcal{G}^{5,3'} = 19998720 \mathcal{G}^{5,P'}; \end{aligned}$$

$$\begin{aligned} N_1^{P^2} &= 1634 \mathcal{G}^{1,3^2} + 11365 \mathcal{G}^{1,4^2} + 13391 \mathcal{G}^{1,6^2} = 26390 \mathcal{G}^{1,P^2} \\ N_2^{P^2} &= 13391 \mathcal{G}^{2,3^2} + 94281 \mathcal{G}^{2,4^2} + 150197 \mathcal{G}^{2,6^2} = 257869 \mathcal{G}^{2,P^2}; \\ N_3^{P^2} &= 150197 \mathcal{G}^{3,3^2} + 809088 \mathcal{G}^{3,4^2} + 1888320 \mathcal{G}^{3,6^2} = 2847605 \mathcal{G}^{3,P^2}; \\ N_4^{P^2} &= 1888320 \mathcal{G}^{4,3^2} + 3870720 \mathcal{G}^{4,4^2} + 19998720 \mathcal{G}^{4,6^2} = 25757760 \mathcal{G}^{4,P^2}; \\ N_5^{P^2} &= 19998720 \mathcal{G}^{5,3^2} = 19998720 \mathcal{G}^{5,P^2} [2]. \end{aligned}$$

References:

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