HYPERBOLIC SYMMETRY AND FRACTALS

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ABSTRACT: Besides all the well known symmetry types that can be generated by a group of isometries, combining congruent images of a motif in Euclidean space, there is another type called "hyperbolic symmetry". This symmetry is obtained by repeating hyperbolically congruent copies of a given motif. Hyperbolic symmetry, together with other interesting mathematical tools, is a basic ingredient in the establishment of a mathematical model to explain fractal and multifractal structures.

1. Hyperbolic Geometry

There are many models of **"hyperbolic geometry"**and we choose **"Poincaré's model"**, that consists in mapping the upper half complex plane $\boldsymbol{H}$ on the interior of Poincaré's unitary disk D

$$\boldsymbol{H} = \{z \in C\colon \operatorname{Im}(z) > 0\} \;\rightarrow\; D = \{\, z \in C\colon \left| z \right| < 1 \,\}$$

In hyperbolic geometry, it is possible to consider **"Fuchsian groups".** A Fuchsian group is a discrete group of symmetries described by 2 x 2 matrices of the form $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, that are isolated from each other in the group of all invertible matrices. Particularly interesting among Fuchsian groups is the so called **"unimodular group"** U, where a, b, c, d are integer numbers and $a\,d - b\,c = 1$. Associated to any discrete group G of symmetries, there is a **"tilling"** of the Universe. And what sounds really exciting is the fact that there are infinitely more tillings of hyperbolic space than from Euclidean space! In Euclidean space you can tile through triangles, squares or hexagons. Instead, in hyperbolic space you can arrange for as many hexagons, for example, to meet at a vertex as you like.

These tillings on the unitary disk are constructed, looking for a **"fundamental region"** in the Fuchsian group. This is a set X such that each transformation in the discrete group G carries X into a disjoint set gX, and that each point of the plane is carried into some point in X by some element of G. Polygonal fundamental regions always exist for Fuchsian groups: there is a natural <u>crystalline</u> structure associated with the group G. Evidently, the tiles get smaller and smaller as we approach Poincaré's disk boundary, that represents the line of infinity. The **"limit set"** of this Fuchsian group can be a self-similar fractal or a multifractal.

To obtain this limit set, we use an **"infinite word structure"** such that longer words code for smaller regions and infinite word structures code for points on the fractal.

Furthermore, given two matrices from the unimodular group such as

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad\qquad Q = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Translation *Inversion*

C. Series has proved [1][2], using continued fraction expansions, that these matrices P and Q generate the entire unimodular group U. In fact, given a real positive number x as a continued fraction expansion such as

$$x = [\, n_1, n_2, n_3, n_4, \cdots]$$

there is an infinite product of generators associated to this number

$$x = P^{n_1}\ Q\ P^{\ n_2}\ Q\ P^{n_3}\ Q\ P^{\ n_4} \cdots$$

In this form, since every real positive number is the limit of a certain sequence of points which lies in some orbit of the unimodular group U, the connection between continued fraction expansions and hyperbolic geometry, becomes evident.

There is a last ingredient missing to complete the recipe: **"Farey tilling"**. To understand how these tilling look like, let us introduce Farey sequence of order n:

$$n = 5 \qquad \frac{0}{1}\ \ \frac{1}{5}\ \ \frac{1}{4}\ \ \frac{1}{3}\ \ \frac{2}{5}\ \ \frac{1}{2}\ \ \frac{3}{5}\ \ \frac{2}{3}\ \ \frac{3}{4}\ \ \frac{4}{5}\ \ \frac{1}{1}.$$

Notice that each fraction is the **"mediant"** of its neighbors $\dfrac{p+p'}{q+q'}$. To get a much more regular order, we introduce the **"Farey tree"**, in which the number of fractions added with each generation is a power of 2. Let us introduce then the **"Farey tilling"** F, that is a tilling by triangles whose vertices line on the x-axis. Reduced fractions of the form p/q , p'/q' are the endpoints of a side of a triangle in F if and only if $\begin{vmatrix} p & p' \\ q & q' \end{vmatrix} = \pm\, 1$. Such Farey triangles can tessellate the entire disk D.

With all these ingredients, M. Piacquadío et al. [3][4], developed a mathematical model to characterize fractal and multifractal sets appearing in physics, biology, chemistry, etc. This characterization is based on the multifractal spectral decomposition technique introduced by I. Procaccia. Using hyperbolic geometry and decomposing the limit sets of finitely generated groups of motions in the Poincaré's circle, the same universal constants obtained in empirical findings are obtained. As we know, for each Fuchsian group, we have a tilling of $\boldsymbol{H}$ or D. The limit set of this tessellation is a **"fractal"** at the infinity line. These fractal structures can be self-similar or not, in which case they are **"multifractals".** This approach has to be related to continued fraction expansions, Farey tillings, the infinite word problem and last but not least, the family of Metallic Means, recently introduced by the author [5][6][7]. Metallic Means are positive irrational numbers enjoying common mathematical properties that are fundamental in analyzing the onset from regular periodic phenomena to chaotic motion. The most paramount of these Metallic Means is the well known Golden Mean ϕ.

Matrices from the unimodular group U can be classified, as is usual, analyzing the sign of the discriminant $\Delta = (d - a)^2 + 4\,bc = (a + d)^2 - 4$. Then,

$$\begin{aligned} &\text{if } \Delta > 0,\ |trace| > 2 : \text{hyperbolic case} \\ &\text{if } \Delta = 0,\ |trace| = 2 : \text{parabolic case} \\ &\text{if } \Delta < 0,\ |trace| < 2 : \text{elliptic case.} \end{aligned}$$

For the limit set $L(G)$ to be a set of positive fractal dimension, we need a minimum of three elliptic generators, while we only need a minimum of two parabolic generators. Such groups are called **"minimally generated groups".** Among parabolic matrices, the simplest ones we choose as generators are the following

$$P = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} = QP^{-1}Q$$

The fundamental region of this minimally generated parabolic group is a rhombus T and it is easy to verify that the limit set of this parabolic group is the whole x-axis (line of infinity). Making a multifractal decomposition of this limit set, Grynberg and Piacquadio [8] found for the minimum value of the concentration

$$\alpha_{\min} = \frac{\ln \sqrt{2}}{\ln \phi} = 0.71,$$

where ϕ is the Golden Mean. This result coincides with empirical results obtained by drawing the spectral curve $(\alpha, f(\alpha))$ for a DLA (diffusion-limited aggregation) grown off lattice in two dimensions. See Procaccia and Zeitak [9].

The infinite word associated with this value of α_{min} in this parabolic group is periodically AP or $(AP)^{\infty}$.

With three elliptic generators that we choose of zero trace, we get instead a limit set that is a fractal of dimension strictly between 0 and 1. This limit set can be visualized as the set of points in the x-axis (line of infinity) that lies inside an infinity of circles, each time smaller and smaller.

In considering multifractal structures, the universal spectral curves have been classified by Tamás Tél [10] in four types I, II, III and IV. And the previously described multifractal algorithm has rediscovered these types,, using three elliptic generators of zero trace. Furthermore, with two hyperbolic generators (minimal number), a new type of universal spectral curve is obtained. It looks symmetrical like type IV but with part of the left leg missing. This sort of spectral curve is the same that has been empirically found by different researchers in quite distinct physical processes. In conclusion, <u>the set of Tél's universal spectral curves is included in the set of spectral curves obtainable by this hyperbolic model.</u>

From the point of view of the infinite word structure, there are large differences between the elliptic and parabolic case on one side and the hyperbolic case on the other. This theoretical result coincide with those obtained by Kohmoto et al. [11], in studying the electronic energy spectrum of a quasi-crystal: the differences in the behavior are due to the fact that $|trace| \leq 2$ corresponds to allowed values of energies, while $|trace| > 2$ corresponds to forbidden energies . Considering the values of α_{min} obtained for the parabolic Fuchsian group $\alpha_{\min} = \dfrac{\ln \sqrt{2}}{\ln \phi}$ and the results for the elliptic one:

$$\alpha_{\min} = \frac{\ln 2}{\ln (a + \sqrt{a^2 + 1})}; \quad \alpha_{\max} = \frac{2 \ln 2}{\ln\left(\dfrac{a + \sqrt{a^4 + 4}}{2}\right)}$$

the question we state is the following:

Is there any condition to be imposed on the value of a ?

The answer is that it must be a positive value no smaller than one. If we analyze consequently the values of the two expressions $a + \sqrt{a^2 + 1}$ and $\dfrac{a + \sqrt{a^4 + 4}}{2}$ for different values of a (not necessarily integers), we arrive to the these interesting conclusions:

- **For any value of a , let it be natural or irrational, the continued fraction expansions of the two expressions, correspond to members of the Metallic Means family.**

- **For a natural, the values of α_{min} for the elliptic Fuchsian group, have purely periodic continued fraction expansions, related to the Golden Mean ϕ and the Silver Mean σ_{Ag} , as well to all its powers.**

- **The values of α_{max} for the elliptic Fuchsian group has periodic and purely periodic continued fraction expansions, related to the Golden Mean ϕ, the Silver Mean σ_{Ag} and the Nickel Mean σ_{Ni} , as well as to all its powers.**

Nevertheless, there is still a galaxy of unexplored cases in this hyperbolic model, e.g. hyperbolic generators, elliptic generators with non-zero trace, non-minimally generated Fuchsian groups, Fuchsian groups with generators of different types and so on, which could explain new roads to chaos, some of them perhaps unexplored.

References

[1] C. Series, "Non-euclidean Geometry, continued fractions and ergodic theory", *The Mathematical Intelligencer,* **4,** Nr 1 (1982).
[2] C. Series, "The modular surface and continued fractions", *J. London Math. Soc. (2)* **31**: 69-80 (1985).
[3] E. Cesaratto, S. Grynberg, R. Hansen and M. Piacquadio, "Multifractal spectra and hyperbolic geometry", *Chaos, Solitons and Fractals* **6**: 75-82 (1995).
[4] S. Grynberg and M. Piacquadio, "Hyperbolic geometry and multifractal spectra", Preprints of the Instituto Argentino de Matemática, Buenos Aires, Argentina (1995).
[5] Vera W. de Spinadel, "On characterization of the onset to chaos", *Chaos, Solitons and Fractals* **8**: 1631-1643 (1997).
[6] Vera W. de Spinadel, "Una nueva familia de números", *Anales de la Sociedad Científica Argentina,* **227**, No. 1 (1997).
[7] Vera W. de Spinadel, "From the Golden Mean to chaos", Editorial Nueva Librería, Buenos Aires, Argentina (1998).
[8] S. Grynberg and M. Piacquadio, "Geometric structure of the Farey-Brocot sequence", *Revista de la Unión Matemática Argentina,* **39**: 209-222 (1995).
[9] I. Procaccia and R. Zeitak, "Shape of fractral growth patterns: exactly solvable models and stability considerations", *Phys. Rev. Lett.* **60**: 2511-2514 (1988).
[10] T. Tél, "Fractals, multifractals and Thermodynamics", *Zeitschr. Naturforsch.* **43a**: 1154-1174 (1988).
[11] M. Kohmoto, B. Sutherland and Chao Tang, "Critical wave functions and a Cantor-set spectrum of a one-dimensional quasi-crystal model", *Phys. Rev. B* **35**: 1020-1033 (1987).