THE CLASSIFICATION AND VISUALIZATION OF PERIODIC TILINGS

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In recent years, much work has been done on the development of the mathematical theory of *Delaney-Dress symbols*, which was inspired by Delaney (1980), suggested in Dress (1984), first applied in Dress & Scharlau (1984) and worked out in more detail in a number of papers including Dress (1987) and Dress & Huson (1987).

This approach associates with every periodic tiling of a simply-connected d-dimensional space a *Delaney-Dress symbol*, which is essentially a finite $(d+1)$-colored graph, in such a way that any two tilings are equivalent (i.e. homeomeric in the terminology of Grünbaum & Shephard (1981)) if and only if the corresponding symbols are equivalent, that is, isomorphic.

Consider the problem of enumerating and visualizing periodic 2-dimensional tilings, i.e. of the euclidean plane, the sphere and the hyperbolic plane. This problem has now been completely solved in the sense that computer programs exist that happily produce tens of thousands of pictures of tilings, solving tasks like: Enumerate all possible distinct (up to homeomerism) tilings of e.g. the euclidean plane, with symmetry group *cm*, and, say, up to 3 classes of tiles with respect to the symmetry group. (There are exactly 12132 such tilings.) See Delgado, Huson & Zamorzaeva (1992), Franz & Huson (1992), and Huson (1992), for details. The speaker intends to demonstrate such a computer program, called *RepTiles*.

The question of classifying periodic 3-dimensional tilings is, of course, of much greater practical importance, but also much more difficult. In a first approach to the classification of periodic 3-dimensional tilings, in Dress, Huson & Molnár (1991) we prove that there exist only finitely many homeomeric types of *face-transitive* periodic tilings of 3-dimensional space. Indeed, for euclidean space we give a complete list of 88 types.

Given a periodic tiling, determining its Delaney-Dress symbol is quite straightforward. The opposite question is: Given an arbitrary, "well-formed" Delaney-Dress symbol, does it really encode some periodic tiling, and if so, of which space (e.g. euclidean, spherical, hyperbolic ...)? For 2-dimensional tilings, Dress (1984) discovered the following, surprisingly simple, condition: Any 2-dimensional Delaney-Dress symbol encodes a tiling of the euclidean plane, the hyperbolic plane, or the sphere, if and only if the *curvature* of the symbol is zero, negative or positive, respectively. In the latter case the symbol must also fulfill the so-called *integrality condition*. The curvature is the sum of certain numbers associated with a Delaney-Dress symbol.

Recently, it has become apparent that the proof of the *curvature criterion* can be based on the classification of 2-dimensional *orbifolds* by Thurston (1978). Indeed, the Delaney-Dress symbol of a tiling can be viewed as a sort of *triangulation*, induced by the tiling, of the orbifold associated with the symmetry group of the tiling. The above integrality condition, then, is what is needed to avoid the four well-known "bad" 2-dimensional orbifolds.

Current research, on the theoretical side, is focused on studying the relationship between 3-dimensional tilings, Delaney-Dress symbols and 3-dimensional orbifolds, aimed at a better understanding of all three concepts. More practically, however, we are working on the concrete enumeration and visualization of periodic tilings of euclidean space. As a consequence of an interdisciplinary research year on *Combinatorics and its Applications* held in 1990-1991 at the *Center for Interdisciplinary Research* at Bielefeld University, we are now cooperating with solid state chemists and crystallographers in England, who are interested in a systematic classification of all (mathematically) possible *zeolite* structures, see Smith (1988).

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