IMPERFECTLY COLORED SYMMETRICAL PATTERNS

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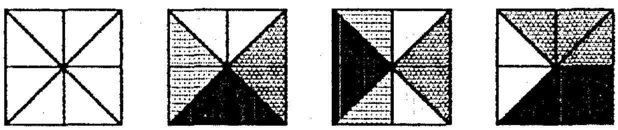
In most studies of colorings of symmetrical patterns, the colorings considered are what are termed perfect colorings. These are colorings in which each symmetry of the associated uncolored pattern effects a permutation of the colors of the pattern. For example, of the colored patterns below obtained from the pattern in Fig. 1-a, the pattern in Fig. 1-b is perfectly colored



(a) (b) (c) (d)

Figure 1.

while the other two are not. The uncolored pattern has eight symmetries, four of which are reflections in the indicated lines through the center of the square and the other four are counterclockwise rotations about the center of the square at angles measuring 90°, 180°, 270°, and 360°. It may be checked that each of these symmetries results in a permutation of the colors in Fig. 1-b. For instance, under a reflection in the horizontal line through the center of the square, colors "black" and "white" are interchanged while the other two colors are fixed. The colored patterns in Fig. 1-c and Fig. 1-d are are not perfectly colored because under a 90° - counterclockwise rotation about the center of the square, color "black" goes to two colors and not just one color.

In this paper, we present a framework for the analysis and creation of colored symmetrical patterns, whether perfectly colored or not. We show the role played by three groups- the symmetry group G of the underlying uncolored pattern, the symmetry group K of the colored pattern (which is the subgroup of elements of G which fix all colors) and the group H of color transformations which consists of the symmetries in G which permute the colors of the pattern. We treat a coloring as a partition of G or a decomposition of G into the union of mutually disjoint non-empty subsets wherein a unique color is assigned to each set in the partition with different sets getting different colors. It is assumed that the uncolored pattern is obtainable as the set of images under symmetries in G of a motif in a fundamental domain for G.

Perfect colorings are colorings where the group of color transformations H is equal to the symmetry group G of the corresponding uncolored pattern. It is well known that a perfect coloring corresponds to a partition of G into left cosets of a subgroup of G. In the general case, where H may be different from G, we show that a coloring corresponds to the partition of G into sets which are unions of right cosets of K, where K is the symmetry group of the associated colored pattern. Using the theory of permutation groups and group actions, we devise a method for determining all colorings of a symmetrical pattern where the subgroups H and K of the group G are specified. For example, if it is specified that the only symmetries of the pattern in Fig. 1-a that will permute the colors are the reflections in the diagonal of the square and the 180° and 360° -rotations about the square's center and with only the 360° -rotation and the reflection in the diagonal sloping upward to the right fixing the colors, we show how to arrive at all the non-equivalent colored patterns satisfying this condition. These are shown in Fig. 2.

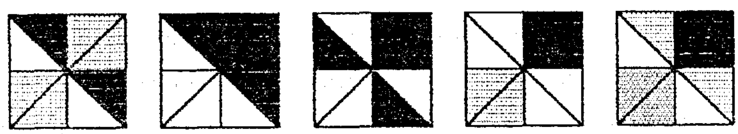


Figure 2.

The method described is applicable to symmetrical patterns in general. In particular, we exhibit illustrations for wallpaper patterns or two-dimensional crystallographic patterns.

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