

EXOTIC TRIANGULAR TILINGS OF THE PLANE – A MODEL FOR NEW APERIODIC CRYSTALS?

Extended summary

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" Exotic triangular tiling " certainly is not a well – defined notion. For instance the PENROSE – tilings were considered quite exotic, when they first became known; and still they are extremely exciting and in my opinion have the strongest aesthetical appeal. Nevertheless meanwhile they became a member of a huge family of tilings that have been widely studied because of their relevance for the theory of " twodimensional " quasicrystals (of course these are threedimensional solid states, but periodic in one dimension and quasiperiodic only in the other two). To every one of these tilings belongs a finite set of prototiles; that is to say : there is a finite collection of paragons, and every tile in the tiling must be congruent to one of the paragons. Furthermore all these tilings can be defined as well by de BRUIJN's strip projection method (in these cases projection from a fourdimensional space onto the plane) as by " inflation rules ". In the simplest case " inflation " means the following : First expand every prototile by the inflation factor η ($\eta > 1$) and then dissect it into pieces, each of which is congruent to some prototile (see fig. 1 for the PENROSE case). By unlimited iteration of of this process one finally gets tilings of the entire plane. Since the inverse procedure – the " deflation " – is unique in these cases, the inflation rules do not permit any periodic tilings. But they guarantee a strong long range orientational order.

In the theory of quasicrystals so far only inflation factors η have played a rôle, which are quadratic algebraic numbers (solutions of an equation of type $ax^2 + bx + c = 0$, where a, b, c are integers). The subject becomes more complicated, when the degree 2 is replaced by any higher number; and these and some others are the " exotic " tilings, I am going to talk about.

I know of at least four infinite series of such tilings :

1) For every natural n ($\neq 1, 2, 3, 4, 5, 6, 8, 10, 12$) the analogue of the PENROSE – tilings, when de BRUIJN's method is used. These tilings seem to lack simple inflation rules and also cannot be defined by local matching rules (for a definition see below).

2) For every odd n not divisible by 3 and larger than 5 the tilings due to K. P. NISCHKE and myself, which are defined by an inflation rule. These tilings hardly can be constructed by de BRUIJN's method and it seems, they also lack local matching rules.

Both these series are based on n -fold rotational symmetry.

3) Tilings of congruent rectangular triangles (one prototile only), defined by inflation with a factor $\sqrt{m^2 + n^2}$, where m and n are coprime (see fig. 2 for $m = 1$, $n = 2$). If the inflation is not the trivial dissection, at least one tile in the first inflation of the prototile is rotated by an angle, which is incommensurate to π . Hence in the global tilings the triangles occur in infinitely many orientations. These tilings cannot be produced by strip projection, nor can they be defined by a local matching rule.

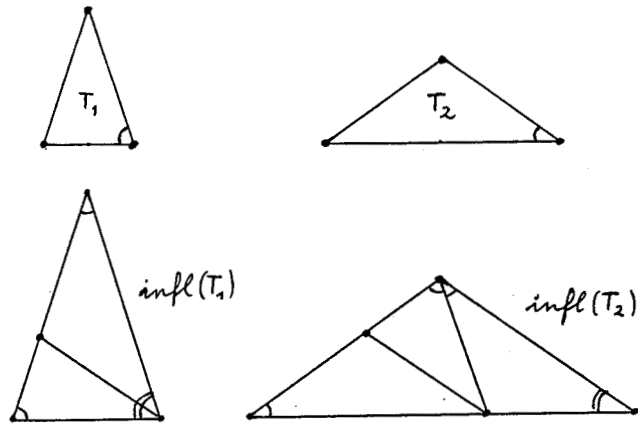
4) A similar construction can be applied on the isosceles triangles with edges of length m and n (see fig. 3).

Besides these infinite series there are quite a few sporadic tilings with exotic properties. I intend to present as well at least one of the planar AMMANN-tilings, as one, where the four prototiles are similar to each other and the inflation factor is $\sqrt{\tau}$ ($\tau := \frac{1}{2}(1 + \sqrt{5})$, the golden ratio), and also a tiling based on sevenfold rotational symmetry defined by inflation and possessing a perfect local matching rule (see figs 4, 5 and 6).

The notion of a local matching rule probably is rather essential for the question, whether a solid state of the corresponding type can exist. Roughly speaking it means: There is an atlas of finitely many bounded patches, and the rule is: Every patch of the tiling, which fits into a circle of a given radius has to be congruent to a member of the atlas.

All the tilings mentioned above are based on appropriate groups of planar symmetries. But most of them do not possess any global symmetry themselves. Some — like the PENROSE-tilings — have local symmetries, which are valid for arbitrary large but bounded regions. In the tiling shown in figure 6 even no small patch obeys a sevenfold symmetry (including the arrows). — Nevertheless all tilings mentioned are "repetitive": To every patch A that fits into a circle of radius r , there is a larger radius R , such that in every circle of radius R a congruent copy of A can be found; and this is true for every value of r .

This is a slightly improved version of the text I sent yesterday by e-mail. Now come the figures; at least some of them may be reduced in size by a higher factor than the text.



The inflation rule for the PENROSE- triangles

Figure 1

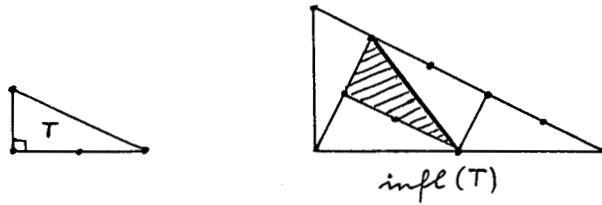


Figure 2

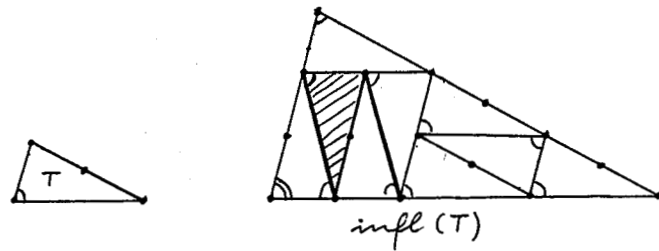
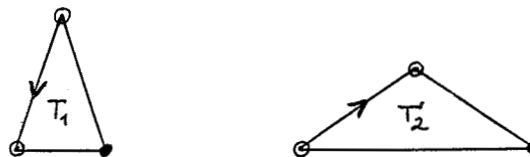


Figure 3



The local matching rule for the PENROSE - triangles : Black vertex must meet black vertex; Arrow must match arrow. (The rule is "perfect")

Figure 4

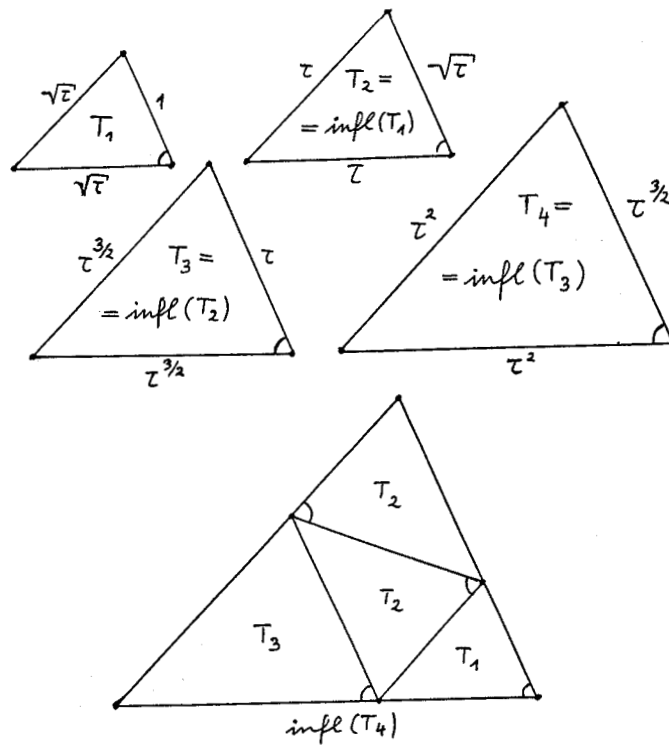
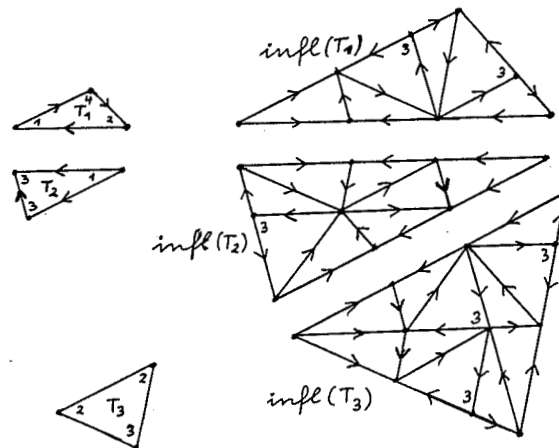


Figure 5



The inflation rule for a tiling based on sevenfold rotational symmetry. The 29 vertexstars can serve as a local matching rule (which also is perfect).

Figure 6