Duality, Nonlinearity, Asymmetry and Symmetry in Lattice Dynamics

Morikazu Toda
Professor emeritus, Tokyo University of Education
Home Address: 5-29-8-108 Yoyogi, Shibuya-ku, Tokyo,Japan

In my study of lattice dynamics, I have been helped by the ideas such as duality, nonlinearity, asymmetry and symmetry. I would like to present how I met these ideas, how they led me to some discoveries, and to see that many theories can be related by these ideas. I was first led to (1)the concept of duality which relates different one-dimensional lattices with the same frequencies of normal modes. In turn, it led me to (2) the discovery of integrable nonlinear lattice with asymmetric interaction potential. Recently I am interested in (3) a lattice with asymmetric potential and asymmetric kinetic energy (symmetric with respect to kinetic and potential energy terms), which describes asymmetric time sequence, such as stochastic process, chemical reaction, and some environment problem.

1) One-dimensional dual lattices

One of the problems I was interested in lattice dynamics was the effect of impurities on the normal mode frequencies (spectrum) of a linear lattice. We sometimes met different lattices wih the same spectra. In this connection, I found the following duality theorem [Toda 1965]:

If we replace masses by springs, and springs by masses, we can obtain another lattice with the same spectrum as that of the original lattice, if certain relations are satisfied between the masses and the force constants of the springs.

These lattices can be symbolically as

A) .....K U K U .....
B) .....U*K*U*K*.....

with kinetic energy terms K,K*, and potential terms U,U*:

$$K_j = \frac{p_j^2}{2m_j} \qquad U_j^* = \frac{k_j}{2}(s_j - s_{j+1})^2$$

$$U_j = \frac{k_j}{2}(x_j - x_{j-1})^2 \qquad K_j^* = \frac{1}{2m_j^*} r_j^2$$

We note that the kinetic energy is symmetric with respect to the momentum p_j or r_j, and the potential energy is symmetric with respect to the relative displacement $x_j - x_{j-1}$ or $s_j - s_{j+1}$. The condition of duality (equivalence) of two lattices A and B is

$$\ldots\ldots m_1 k_1^* = m_1^* k_1 = m_2 k_2^* = m_2^* k_2 = \ldots\ldots$$

The simplest equivalence of dual lattices can be achieved by the replacement

$$\frac{1}{m_j} = k_j^*, \quad \frac{1}{k_j} = m_j^*,$$

$$p_j = s_j - s_{j+1}, \quad x_j - x_{j-1} = r_j \quad .$$

2) Nonlinear lattice

The idea of the dual lattice was extended to nonlinear lattices [Toda 1966]. For simplicity we consider a uniform lattice with the kinetic energy term $K_j = p_j^2/2m$, and the potential energy term

$$\hat{U}_j = \phi(x_j - x_{j-1}).$$

the corresponding dual lattice has the potential energy term $U_j^* = (s_j - s_{j+1})^2/2m$, and the kinetic energy term

$$\hat{K}_j^* = \phi(r_j)$$

^ means ninlinear. $\hat{K}$ is a kind of kinetic energy with a mass which depends on the momentum r_j of the lattice B. We see that the lattices A and B symbolically expressed as

A) .....K $\hat{U}$ K $\hat{U}$ K .....

B) .....U*$\hat{K}$*U*$\hat{K}$*U*.....

are equivalent (behave the same).

Because the lattice B seemed simpler than the lattice A, I worked on B, and was able to find an integrable lattice [Toda 1967].

This lattice has the asymmetric potential

$$\phi(r)= e^{-p}-1+p$$

It was found that this nonlinear lattice is perfectly integrable [Lunsford and Ford 1972, Hénon 1974, Flaschka 1974].

3) Lattice asymmetric in time and space

Recently I am interested in a lattice with asymmetric kinetic energy as well as asymmetric potential energy. This lattice can be expressed symbolically as

$$.....\hat{K}\ \hat{U}\ \hat{K}\ \hat{U}\ \hat{K}\ .....$$

The dual lattice is the same to the original lattice (self-dual). Motion in this lattice is asymmetric in time and space. That is, the motion to the right is different from the motion to the left. This strange behavior comes from the asymmetricity of the kinetic energy term.

Further, it turned out that this lattice is equivalent to a system considered by Kac and Moerbeke [Kac and Moerbeke 1975] as a model for a certain stochastic process, and to a special case of the Lotka-Volterra model [Lotka 1956, Volterra 1931] for cofl-icting species, or environment problems.

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ROTARY SHADOWS FROM THE p-DIMENSIONAL HYPERSPACE

M. Torres*, G. Pastor, I. Jiménez** and J. Fayos*****

* Instituto de Ciencia de Materiales, Sede A,

** Instituto de Electrónica de Comunicaciones,

*** Instituto de Química-Física "Rocasolano",

Consejo Superior de Investigaciones Científicas,

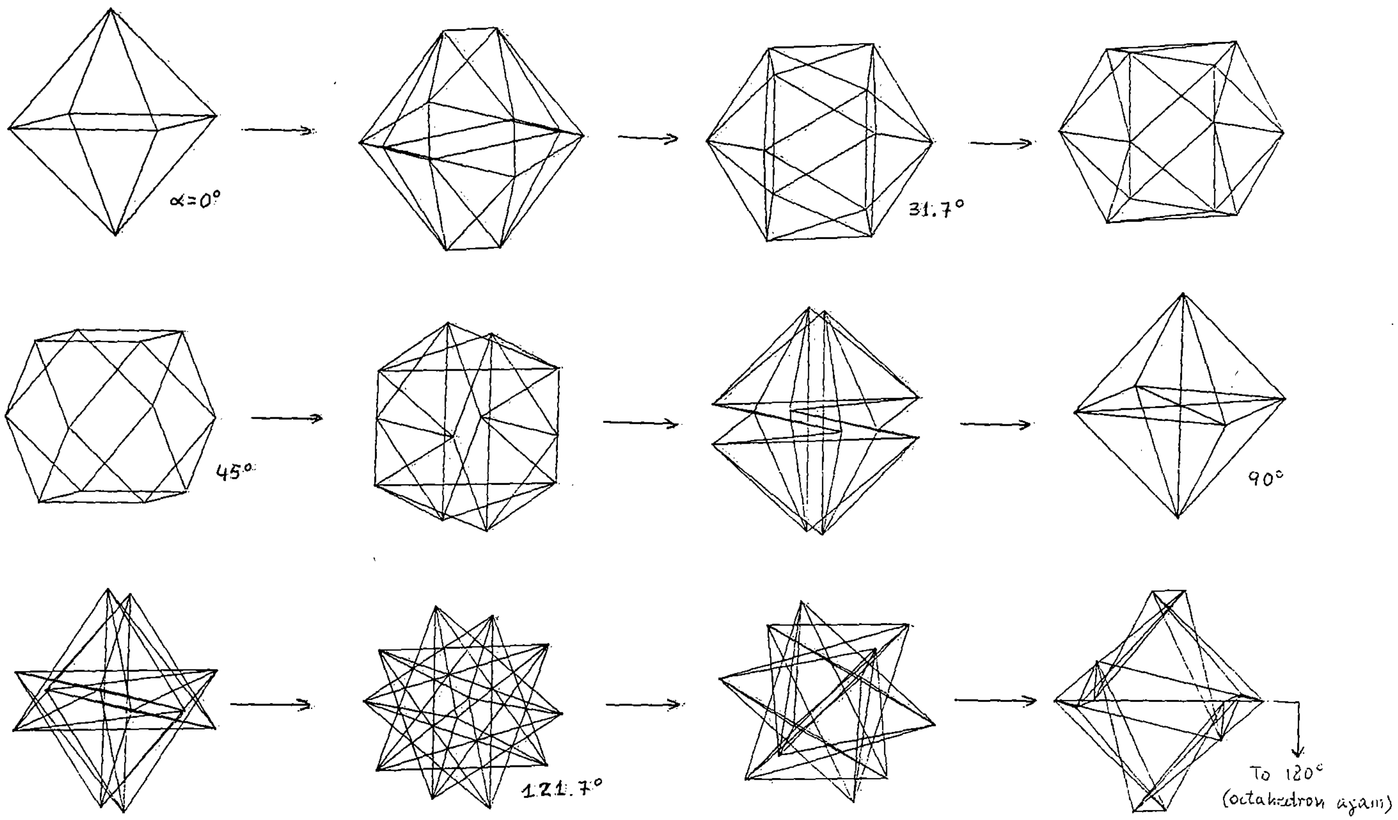
Serrano 144, 28006 Madrid, Spain.

In a recent paper, we found some continuous evolutions that connect the radial skeletons (center to vertices directions) of basic polyhedral forms [1]. Afterward, taking into account the works of Hadwiger and Coxeter [2] about hypercrosses shadows falling on the ordinary spaces E^2 and E^3, we have extended our above mentioned work in order to connect in a continuous way the icosahedral and cubic symmetries and orders [3,4].

Here, we use our "rotary shadow method" [4] to generate two appealing geometric evolutions. We begin finding variable vectors half-stars in a rotary subspace E^n which represent the orthogonal projections of half-crosses defined in the hyperspace E^p, p>n. So, according to the theorem of Hadwiger [2], we start from variable half-stars with p vectors which preserve the equation

$$\sum_{i=1}^{p} v_{i\mu} v_{i\gamma} = \frac{1}{n} \delta_{\mu\gamma} \sum_{i=1}^{p} v_i^2 \quad ; \mu,\gamma = 1, \ldots, n \quad , \qquad (1)$$

where $(v_{i1}, v_{i2}, \ldots, v_{in})_{i=1,\ldots,p}$ are the components of the p



$\alpha = 0°$
31.7°
45°
90°
121.7°
To 180°
(octahedron again)

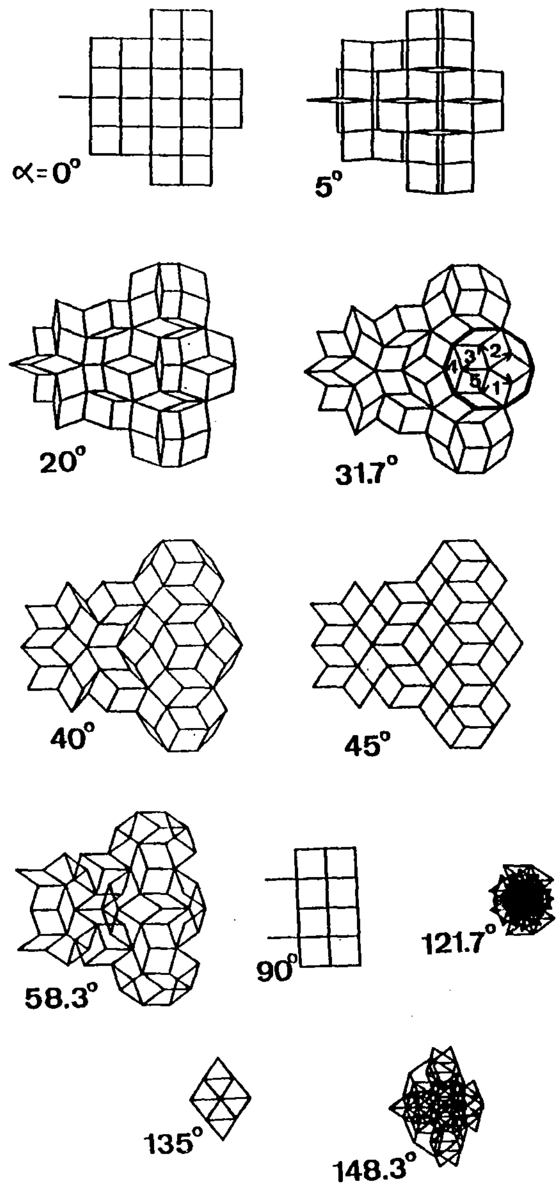


Fig. 2

vectors $\{v_i\}_{i=1,\ldots,p}$ in the rotary subspace E^n.

The α-variable half-star $v_1=(c,-s,0)$, $v_2=(c,s,0)$, $v_3=(0,c,-s)$, $v_4=(0,c,s)$, $v_5=(-s,0,c)$, $v_6=(s,0,c)$, where $c=\cos\alpha$ and $s=\sin\alpha$, preserves eq.1 (with p=6 and n=3). By drawing an adequate set of edges which join the vertices of this α-variable star, the octahedron metamorphosis can be seen. Fig.1, where star vectors are omitted, schematically shows this cycle which connects in a continuous way some notable polyhedra (octahedron, icosahedron, cuboctahedron and small stellated dodecahedron)

The α-variable half-star $v_1^*=(c^2,-s)$, $v_2^*=(c^2,s)$, $v_3^*=(-s^2,c)$, $v_4^*=(-2sc,0)$, $v_5^*=(-s^2,-c)$ preserves eq.1 (with p=5 and n=2). When $\alpha=31.7174°$, this half-star coincides with the five pentagonal directions of a Penrose tiling [5]. When α varies, the half-star and the Penrose tiling also evolve in a continuous way. Fig.2 shows an evolutionary cycle of a Penrose tiling patch. This cycle contains a singular state of extreme folding or reversible collapse ($\alpha=121.7174°$).

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