Costa Minimal Surfaces: D_4 Symmetry Sculpture by Virtual Image Projection, Part II

Sam Ferguson
University of Michigan, Ann Arbor, Michigan

Helaman Ferguson
Supercomputing Research Center, Bowie, Maryland
Email: helamanf@super.org

Alfred Gray
University of Maryland, College Park, Maryland

Costa's minimal surface is a triply punctured torus and tori relate to elliptic curves and thence to elliptic functions. Elliptic functions come in families, the specific ones important here are those of Weierstrass, the 'pe', the 'zeta', the 'sigma' which we denote by the letters P, Z and S as they pertain to the unit square or the lattice of Gaussian integers or the lemniscatic case. The D_4 symmetry proof requires the properties of the P and Z functions under changes in argument w to $-w, iw, \bar{w}, w+m+in$, where m, n are integers. The derivative of the Z function is the negative of the P function, $Z' = -P$. The derivative of the P function, P', is related to P by the cubic equation $(P')^2 = 4P(P^2 - P(\frac{1}{2})^2)$. The function S is an entire function of a complex variable $w = u + iv$ defined as a product over the lattice of Gaussian integers. The P, P', and Z functions can be written in terms of this S function.

The parametrization for the Costa minimal surface takes two variables (u, v) into three coordinates (x, y, z). The three coordinates and the normal vector to the surface can be given in terms of the following three functions of a complex variable $w = u + iv$.

$$\alpha(w) = \frac{\pi^2}{8\lambda^2} +$$

$$\frac{1}{2}\Re\Big(\pi w - Z(w) + \frac{\pi}{2\lambda^2}\big(Z(w - 1/2) - Z(w - i/2)\big)\Big)$$

$$\beta(w) = \frac{\sqrt{2\pi}}{4}\log\left|\frac{P(w) - \lambda^2}{P(w) + \lambda^2}\right|$$

$$\gamma(w) = \frac{2\sqrt{2\pi}\lambda^2}{P'(w)}$$

Then

$$(x, y, z) \leftrightarrow (\alpha(w), \alpha(i\bar{w}), \beta(w))$$

are the coordinate functions and

$$\frac{(2\Re\gamma(w), 2\Im\gamma(w), |\gamma(w)|^2 - 1)}{|\gamma(w)|^2 + 1}$$

is the corresponding normal vector.

The number $\pi = 3.14159265358979323846\ldots$ is the familiar area of a circle of radius one with equation $x^2 + y^2 = 1$. The number $\lambda = 2.62205755429211981046\ldots$ is related to the less familiar lemniscatic curve with equation $x^4 + y^2 = 1$. λ is related to π and the Euler gamma function by $\lambda = \frac{1}{4}\Gamma(\frac{1}{4})^2\sqrt{\frac{2}{\pi}}$, also, $\lambda^2 = P(1/2)$. The choice of exactly λ gives an embedding, i.e., no self-intersections in the minimal surface.

The D_4 symmetry is crucial for the construction of a Costa sculpture out of eight symmetric parts. D_4 is the formal symbol for the group of symmetries of a square. The Costa minimal surface is a triply punctured torus with fundamental domain a unit square. The punctures correspond to the three points $(0,0), (\frac{1}{2},0), (0,\frac{1}{2})$ which are actual poles of the coordinate functions. The other five points are opposite side torus identifications of these. E.g., $(1,0), (1,1), (0,1)$ all identify with $(0,0)$, whereas $(\frac{1}{2},1)$ is identified with $(\frac{1}{2},0)$ and $(1,\frac{1}{2})$ is identified with $(0,\frac{1}{2})$. In the image of the surface in three dimensions these eight poles correspond to points at infinity, whereas the center of the unit square $(\frac{1}{2},\frac{1}{2})$ will correspond to the center of symmetry of the surface.

These eight transformations of D_4 can be written algebraically as $t_0, t_1, t_2, t_3, t_4, t_5, t_6, t_7$ maps (u,v) to $(u,v), (v,u), (1-v,u), (1-u,v), (1-u,1-v), (1-v,1-u), (v,1-u), (u,1-v)$, respectively. Because of the double periodicity of the triple $(x,y,z) = (c_1,c_2,c_3)$ we restrict $w = u + iv$ to the fundamental square defined by $0 < u < 1$ and $0 < v < 1$. Because of the D_4 symmetry we can only need to compute the vector (c_1,c_2,c_3) as a function of the two variables u,v in the fundamental triangle (bounded by East and Northeast) defined by the inequalities $0 < v < u < 1$. Including rotations *and* reflexions the surface is defined by its part in one single octant, but reflexions are not physical! Another problem is that the P and Z functions have poles; this leads to a natural pentagon as an eighth of a finite part of the surface.

For the proof, there are eight transforms of the complex number $w = u + iv$ which correspond exactly to the u, v transforms above. $w, (x, y, z)$; $i\bar{w}, (y, x, -z)$; $1 + iw, (-y, x, -z)$; $1 - \bar{w}, (-x, y, z)$; $1 + i - w, (-x, -y, z)$; $1 + i - i\bar{w}, (-y, -x, -z)$; $i - iw, (y, -x, -z)$; $i + \bar{w}, (x, -y, z)$. Transformations t_1 and t_3 generate the group, $t_2 = t_1 t_3$, $t_4 = t_1 t_3 t_1 t_3$, $t_5 = t_3 t_1 t_3, t_6 = t_3 t_1$, $t_7 = t_1 t_3 t_1$, where $t_0 =$ identity.

The stereo view above is a fragment of a fundamental eighth of the Costa minimal surface, e.g., t_0 alone. The bronze and aluminum Costa II and the molds for Costa III were made from VIP quantified forms so that the D_4 symmetry was part of the construction. The natural stone of Costa V and Costa VI, exhibits the D_4 symmetry but is carved out of the natural stone directly with the fast global algorithms (which use the D_4 symmetry!) in the NIST VIP SP–2 system.

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