**Costa Minimal Surfaces: $D_4$ Symmetry Sculpture by Virtual Image Projection, Part I**

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The Costa minimal surface sculptures are a good example of applied mathematics come full circle over developments in the last two centuries: make observations from nature about soap films (Plateau), write down differential equations describing area minimizing surfaces (Euler-Lagrange), define a minimal surface geometrically in terms of curvature (Gauss), discover a minimal surface with non-trivial topology (Costa), draw computer images of the surface (Hoffman-Hoffman), recognize symmetry and prove the surface has no self-intersections (Hoffman-Meeks), discover fast parametric equations for the surface (Gray-Ferguson), and finally return to nature with a sculpture, a touchable form of a 'soap film'. The CRADA between NIST and Helaman Ferguson's studio is a vehicle for applying new technological tools to create aesthetic objects. One current application of this technology is to create a ten foot diameter minimal surface Costa IV. This paper includes a summary of techniques for creating the 16 inch diameter bronze and aluminum piece (Costa II) and a three foot diameter fiberglass piece (Costa III) and the ten foot diameter Costa IV, the Carrara marble Costa V, the ten inch diameter Carrara marble Costa VI, ... .

Thirty of Helaman Ferguson's recent sculptures are photographed and documented in [3]. If some things are expressible in a mathematical language it may be possible to express them naturally in a computer graphical context. On the other hand, there is nothing intrinsically sculptural about mathematics or computer science in general, any relationship is a creative act in itself. Geometry and topology tend to be a little closer to the bones of sculptural intent. Helaman Ferguson carved *Umbilic Torus NC* using a CNC milling machine with a surface filling curve tool path, [4]. By contrast he carved *Umbilic Torus NIST* directly in stone using the 'virtual image projector' VIP SP-1, [5]. A Mac-II computer monitored lengths of three cables under tension which met at a point. The virtual image for this sculpture was in the form of specially designed parametric equations. The software included an algorithm which calculated from any given point on the block the nearest distance from that point to the virtual image.

The second generation of the virtual image projector, SP–2, built by the Robot Systems Group at NIST, the National Institute of Standards and Technology, Gaithersburg, Maryland, was used in the creation of *Eight Fold Way*, cf., [3]. SP–2 has six instead of three cables with all six lengths monitored by sensors arranged in Stewart platform format, [1]. The operator interactively flies the triangle with attached tools. Tool tip position coordinates $(x, y, z)$ and tool orientation (pitch, roll, yaw) are computed from the six cable lengths. Carving the *Eight Fold Way* included matching two stone parts with the help of the VIP SP-2. The precise serpentine geometry of the underlying disc for the *Eight Fold Way* counterpoints the free marble topology. The 232 stones forming a natural heptagon tessellation of the Poincare disc were cut to within 3 thousandths of an inch with a computer driven water jet, a filament of water issuing from a diamond orifice under 55,000 pounds per square inch pressure, cf. [6].

The book by Alfred Gray, [7], is filled with images of mathematical surfaces which took many person years to create before computers. Now images appear on the computer screen in seconds with a few keystrokes by anyone who can type. Explicit quantitative sculpture includes a quantitative (mathematical) creation prior to the physical creation. The physical artifact then partakes in various ways of the original quantitative creation, but can be convolved with geologically or physically interesting natural materials. Technology is just emerging to make such sculpture possible in person hours instead of months or years. But it should be kept in mind, *even possible at all*, due to the huge numbers of calculations involved.

VIP or Virtual Image Projection is an *inverse digitization* process The current software includes a C language implementation of this model which takes the six lengths imput and computes six coordinates which are three for the location of the tool tip and three for the orientation of the tool. The Costa surface involves undercuts so computing the orientation of the tool in the undercut spaces is necessary. The following stereo pair of the VIP octahedron shows the lower triangle translated downward with yaw.

The theory of minimal surfaces emerged two centuries ago, when mathematicians Euler and Lagrange began thinking about mathematical language to describe stretched surfaces, surface tension, soap films, and such. The abstract mathematical surfaces were called minimal surfaces and were defined by certain differential equations, cf., Osserman [9]. At first three surfaces were discovered, the plane, helicoid, and catenoid, which satisfied the differential equations for minimal surfaces. And there the matter lay for two centuries, with various mathematicians wondering if there could be a minimal surface with a hole in it. Costa's thesis [2] led to a renaissance in examples of complete embedded minimal surfaces of finite genus, an opportunity immediately seized by Hoffman and Meeks [8] who have generalized Costa's inspiration from one handle and three ends to many handles and many ends. Helaman Ferguson celebrates these centuries of development with his Costa I-IV-... series of minimal surface sculptures.

Perhaps potato chips are the most common example of a minimal surface. A potato chip starts out as a flat thin slice of a moist potato. It then gets dried by frying and undergoes shrinkage or area minimizing. The resulting saddle or hyperboloid, the familiar potato chip form, is a natural minimal surface with boundary. The potato chip surface $t = k(u^2 - v^2)$ has mean curvature zero on the two diagonals $v = \pm u$, in particular is minimal at the origin where the gaussian curvature is $-k^2$. This potato chip surface is unchanged under the transformation $t_1 : (u, v, t) \mapsto (v, u, -t)$. This transformation will appear as a fundamental symmetry of the Costa minimal surface. Such potato chips match minimal surfaces to second order.

$D_4$ is the symmetry group of a square, there are eight elements which describe all the rotations and reflexions. Costa's minimal surface has $D_4$ symmetry but this was not remotely guessed in 1983 when Costa wrote down the parametric equations [2]. As soon as an even rough computer graphical image was computed by James Hoffman and seen by David Hoffman and William Meeks, the symmetry was exposed. From there a proof was constructed [8]. Costa's original parametric equations involved the direct Weierstrass representation which integration of functions of the Weierstrass elliptic $P$ function. Recently Alfred Gray carried out the integration symbolically in terms of another Weierstrass elliptic function $Z$. Furthermore Gray stimulated the inclusion of the Weierstrass elliptic $Z$ function in the Mathematica 2.3 package Elliptic.m, [10]. This breakthrough in efficiency meant the computation of a picture of the Costa minimal surface took five minutes instead of ten hours. Now with the new $S$ function algorithms in C a whole picture takes a few seconds to compute. This renders the surface feasible in my virtual image studio context (Mac-II). A year ago this sculpture could not have been carved in real time. Gray's original parametrization was reasonable to use for the 16 inch Costa II sculpture; it was not suitable for precise direct carving of stone. We have written C implementations for finding both the nearest point and nearest point along a tool direction to the virtual image, a key computational component of the VIP system. This allows real time information at the air drill tip even on the Mac-II.