Quasiperiodic Patterns and Golden Sections.

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In this lecture we review the transition from periodic to quasiperiodic patterns, describe the new features of quasiperiodic structure and its links to the golden section, and illustrate the appearance of these patterns in the new physics of quasicrystals.

1. The appearance of periodicity:
The most familiar roots for the notion of periods is our experience of time. Time is experienced in terms of units like days, months, years. These units or periods follow each other in a sequential repetition pattern. With the expansion of experience and science, we have learned about the astronomical origin of these and other much longer periods, and we have also found extremely short periods of time on the atomic and subatomic scale. Among the arts, music is a prominent field where a pattern in time is created, very often with a background of periodic rhythmic units.
Periods in space were created, a long time before science, in the sequential order of band ornaments, and the additional dimensions of space allowed for the extension of periodic structures to ornaments in the plane. The inclusion of new dimensions opened the way not only to doubly periodic patterns, but also brought the new repetion patterns of rotational, mirror and color type. The richness of ornamentics is obtained from the variation of motives and from their combination. In the structure of crystals one finds up to triply periodic structures from the macroscopic down to the atomic level. The motives of this periodic structure are formed from atoms and have as their typical length scale multiples of an Angstrom.
The systematic description of these periodic patterns in science has a long history up to present-day crystallography.

2. The structure of periodic patterns:
A simple way of representing sequential periodic patterns is given by forming artificial words, that is, sequences from an alphabet of letters a,b,c,.... As an example consider the words cde, cdecde, cdecdecde,....
which are generated recursively from the motive cde. In the limit of an infinite number of repetitions we get a periodic sequence which is very similar to a periodic decimal fraction.
In mathematics we describe the periodic property of this infinite sequence by a shift map. This shift map moves each motive cde by three letters to the right, so that the unshifted and shifted patterns are
...cdecdecde...... ,
......cdecdecde... .
Since the points are meant to represent the infinite sequence, the unshifted and shifted sequences represent the same pattern, and so the shift map generates a symmetry map of the sequence. The elaboration of the possible repetition patterns for multiply periodic patterns in two- and three-space is the subject of crystallography.

The notion of various types of symmetries, of maps and motives and their interplay in art and science will be illustrated by selected examples.

3. The structure of quasiperiodic patterns:
To introduce the concept of quasiperiodic patterns, we take as an example the Fibonacci words formed recursively from two letters a, b:
a, b, ba, bab, babba, babbabab, ...
To continue this sequence recursively, one simply links to the right of the last word its precessor. The infinite sequence obtained in this fashion is called the Fibonacci sequence. If one counts the frequency of the letters a and b in the sequence, one finds subsequent pairs of the well-known Fibonacci numbers
1, 1, 2, 3, 5, ...
From the relation of these numbers to the golden section it is easy to demonstrate that the Fibonacci sequence cannot be a periodic sequence based on a finite motive formed from the letters a and b. So this sequence is an example of a so-called quasiperiodic object generated by a strict recursive rule. Patterns formed in a similar fashion by words are studied in the new field of mathematics called Combinatorics on Words, compare Lothaire (1983).
Now let us replace the letters a,b by two general motives to obtain the prototype of a quasiperiodic structure. Quasiperiodic patterns are not restricted to one-dimensional sequences, and richer patterns are obtained in two and three dimensions. Moreover, these quasiperiodic patterns may display rotational symmetries which are incompatible with all the well-known periodic structures. Patterns in the plane of this type with fascinating properties were invented by Penrose (1974). In three dimensions, similar patterns were proposed by Mackay (1982) and analyzed by Kramer and Neri (1984).
Properties and principles for the generation of these and similar patterns will be discussed and illustrated. The new interpretation of quasiperiodic patterns as sections through periodic patterns in hyperspace will be elaborated. Recently it has been realized that Kepler (1611) anticipated some important concepts of quasiperiodic structure. The significance of Keplers work was recognized already by Kowalewski (1938).

4. The golden section:
From the numerous ramifications of the golden section in the arts, in mathematics and science, as described for example by Huntley (1970) and by Richter and Scholz (1987), we select and discuss its relation to geometry and in particular to pentagonal and icosahedral symmetry. These symmetries necessarily are in conflict with periodic symmetry and hence were excluded from traditional crystallography, as will be illustrated by examples. The unexpected new result is that the golden section can be linked to quasiperiodic patterns rather than to periodic ones.

The connection between the golden section and quasiperiodic symmetry will be discussed in detail, starting from the Fibonacci sequence which is described above. As a result, quasiperiodic patterns with pentagonal and icosahedral symmetry will be interpreted as golden sections through periodic patterns in hyperspace. Periodic patterns formed from hypercubes as described by Haase et al. (198 ) will serve as a main example. Sections through these hypercubes generate families of cells in three-dimensional space known in part as zonohedra. These cells fit together in a subtle way to form quasiperiodic fillings of three-space.

5. Quasicrystals:

Experimental evidence for the existence of quasiperiodic structures formed by metal alloys was given first by Shechtman et al.(1984). Not only the study of these materials by X-ray and electron diffraction, but also the outer morphology of these quasicrystals given by Dubost et al. (1986) shows that icosahedral symmetry and hence the golden section is manifest in these alloys. Only part of the physics of quasicrystals has been explored and understood up to now. A non-technical survey of the present state of this field will be given, with emphasis on the evidence for a quasiperiodic cell structure.

For a long time, the golden section has played its main role in the non-scientific fields of culture. With the new role of the golden section played in quasicrystals, there is a much better chance of fruitful explorations from science and other fields of culture into the fascinating new landscape of quasiperiodicity.

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