SYMMETRY GROUPS IN MATHEMATICS, ARCHITECTURE AND ART

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Fields of interest: Fractal Morphology, Onset to Chaos, Number Theory (Continued Fraction Expansions of Real Numbers).

Awards: Several national Research and Development prizes.

Publications and/or *Exhibitions:*

1) "The Metallic Means family and forbidden symmetries", International Mathematical Journal, vol. 2, Nr. 3, pp. 279-288, 2002;

2) Exhibition "Fractal Art", Fifth Interdisciplinary Symmetry Congress and Exhibition of the ISIS-Symmetry, July 14, 2001, Sydney, Australia,

3) "Continued fraction expansions and Design", Proc. of Mathematics & Design 2001 – The third International Conference, July 3-5 2001, Deakin University, Geelong, Australia;

4) "The family of Metallic Means", Visual Mathematics, vol. I, Nr. 3, 1999. http://members.tripod.com/vismath/;

5) "From the Golden Mean to Chaos", book edited by Nueva Librería, ISBN 950-43-9329-1, 1998.

Abstract: *The word "symmetry" has two meanings. A symmetric object is well proportioned but the concept is not restricted to concrete objects; the synonym "harmony" refers to its use in Acoustics and Music. The second meaning is that of the geometric bilateral symmetry, the symmetry so evident in superior animals, especially in men. From the mathematical point of view, a whole symmetry theory can be considered for applications. From this theory, we have chosen the "symmetry group of the square" to present interesting uses in Architecture and art.*

1. Symmetry in Culture

The word symmetry comes from the Greek symmetria, meaning "the right proportion". From the historical point of view, the term symmetry has denoted many meanings, depending on the field of human knowledge where it was used. Notwithstanding, "symmetry is a unifying concept", as Hargittai's, Magdolna and István, have proved in their beautiful and unique book "Symmetry". Indeed, the concept of symmetry can provide a connecting link among many different fields of endeavor, perhaps the best and more appropriate link to protect human studies from the increasing and separating compartmentalization within our scientific world.

Going back to the year 27 B.C., we found a monumental work: the 10 books written by the roman architecture Vitruvio (probably Marco Vitruvio Pollione) and dedicated to the Emperor August Architecture, says Vitruvio, depends from order, disposition, eurhythmy, property, symmetry and economy. These terms have today, completely different meanings. E.g., order, says Vitruvio, confers the appropriate measure to the elements of a certain building, when considered separately and symmetry, gives concordance to the proportions of the different parts of the construction. This approach to the meaning of symmetry is quite similar to the mutually corresponding arrangement of the various parts of a human body around a central axis, producing a proportioned balanced form.

Vitruvio dedicated much time to the study of the proportions in the human body in his considerations on symmetry. *Symmetry*, says Vitruvio, *comes from proportion, that is, from a correspondence between the dimensions of the parts of a whole and of the whole with respect to a certain part selected as a model, the module*. Such a selection of parts of the human body as a module, initiated probably by Vitruvio, was the very beginning of a historical ergonomic chain linking Vitruvio, Albrecht Dürer, Leonardo da Vinci and many, many other artists, including the modern contemporary architect Le Corbusier.

2. Symmetry in Mathematics

In Euclidean geometry, a *"symmetry"* of a figure is a rigid motion that leaves the figure unchanged. And what is a rigid motion? A *"rigid motion"* of the plane is any way of moving all the points in the plane such that

- The distance between points stays the same
- The relative position of the points stays the same

This concept of symmetry in the plane is easily generalized to symmetry in three-dimensional space.

The simplest rigid motion is *"translation"*. In a translation, everything is moved by the same amount and in the same direction. We specify a translation by drawing an arrow. Another rigid motion of the plane is *"rotation"*. A rotation fixes one point and everything rotates by the same angle around that point. A third rigid motion is *"reflection"*. A mirror line determines a reflection and it is easy to prove that

- points on the mirror line are unchanged by reflection;
- the distance from a point to the mirror is the same as the distance from the image of that point to the mirror.

Finally, combining reflection with translation, we get a *"glide reflection"*, that is, a mirror reflection followed by a translation parallel to the mirror. To make the list complete, we add the *"do-nothing"* operation as another rigid motion of the plane.

All these rigid motions of the plane generate different types of symmetries:

Translation symmetry or repetition: it means shifting and repeating the same motif, producing a *"periodic"* pattern. In ornamental art, this type of symmetry is called *"infinite ratio"*, in which case it is a glide reflection, like in the frieze of Persian archers at Dare's Palace, in Susa (at present Khuzistan, Iran), depicted in Fig. 2.1 or the Palace of the Doje in Venice, Italy (Fig. 2.2).


Fig. 2.1


Fig. 2.2

Rotational symmetry: it is an operation that iterated, brings the configuration again to its original position. To avoid ambiguity, it is normally assumed that counterclockwise rotations are positive. On the plane, the simplest figures that exhibit rotational symmetry are regular polygons. In three-dimensional space, a body has rotational symmetry with respect to an axis r if every rotation around r, takes the configuration back to its original position. E.g., a four-blade pinwheel has rotational symmetry and if all its petals are of different colors, it is easy to prove that after a whole turn, we are back at the starting position. This is called a *"4-fold rotational symmetry"*.

Mirror symmetry or bilateral symmetry: it occurs when two halves of a whole are each other's mirror images. This symmetry was also called *"heraldic symmetry"*, because the old inhabitants of Sumer (southern part of Mesopotamia) were the first ancient people using heraldic drawings, having its use being extended later on to Persia, Syria and Byzance.

To introduce a mathematical language and redefine symmetry, let us consider a spatial configuration F. We shall say that the motions that leave F invariant -- or unchanged --, form an algebraic structure called a *"group of transformations"* G, defined in the following way:

Let G be a set together with a composition law $\circ$ which associates to each pair $g,h \in G$ another element $g \circ h \in G$, called the *"product"* of g and h. Suppose that this product satisfies the properties:

a) $I \in G$ (the identity is in G)

b) if $s \in G$ then $s^{-1} \in G$ (the inverse transformation is in G)

c) if s and $t \in G$ then the composition $s \circ t \in G$.

Then the set of transformations G forms a *"group"*.

This group of transformations describes exactly the symmetries of the figure F.

To describe three-dimensional space, the notion of *"congruence"* is very useful. Two spatial regions are congruent if a rigid body in two different positions can occupy them. Congruent transformations form a group and, obviously, the simplest types of congruencies are translations, reflections and rotations. The symmetry of any figure in three-dimensional space is described by a subgroup of the group G.

Example 1: Let us consider the symmetries of a square. The eight distinguishable spatial transformations, which comprise this group, are four quarter-turns and four reflections (see Fig. 2.3):

d_1 : Do-nothing
d_2 : Rotation by 1/4-turn
d_3 : Rotation by 1/2-turn
d_4 : Rotation by 3/4-turn

d_5 : Reflection in mirror m_1
d_6 : Reflection in mirror m_2
d_7 : Reflection in mirror m_3
d_8 : Reflection in mirror m_4

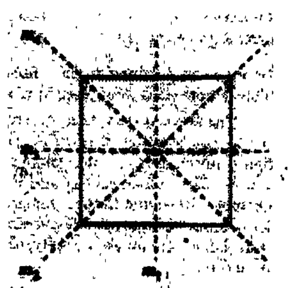

Fig. 2.3

By direct inspection of the figure, it is evident that a square has four rotation symmetries and four reflection symmetries. This is a property that can be extended to all regular polygons, and it is easy to prove the following general result: *The regular polygon with n sides, has exactly n rotation symmetries and n reflection symmetries.*

Adopting the usual composition of transformations as the fundamental operation, we may write the composition table called Cayley diagram, that gives the results of the composition $d_j \circ d_k$, for $j = 1,...,8$; $k = 1,...,8$ and describes the group of symmetries of a square.

					j				
		1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
	1	1	2	3	4	5	6	7	8
	2	2	3	4	1	8	7	5	6
	3	3	4	1	2	6	5	8	7
k	**4**	4	1	2	3	7	8	6	5
	5	5	7	6	8	1	3	2	4
	6	6	8	5	7	3	1	4	2
	7	7	6	8	5	4	2	1	3
	8	8	5	7	6	2	4	3	1

The symmetries of a square form a group (non-commutative, as is easily verified by the non-symmetrical table) and this result is generalized in the following sense: The collection of symmetries of any figure will always be a group.

Furthermore, it is easy to prove that

$$d_3 = d_2^2 \,;\; d_4 = d_2^3 \,;\; d_6 = d_2^2 \circ d_5 \,,\; d_7 = d_2 \circ d_5 \,;\; d_8 = d_2^3 \circ d_5 \,.$$

Notice that the elements d_6 , d_7 and d_8 can be expressed by the compositions of d_2 and d_5 alone

Example 2: Another interesting example is the famous pentagonal star with which Faust exorcised Mephistopheles (Fig. 2.4) It coincides with itself by performing 5 rotations of angles 72°, 144°, 216°, 288°, 360° and the 5 reflections with respect to the lines linking the center of the figure with the 5 vertices of the pentagon. These 10 operations form a group and this group indicates what sort of symmetry this star has.

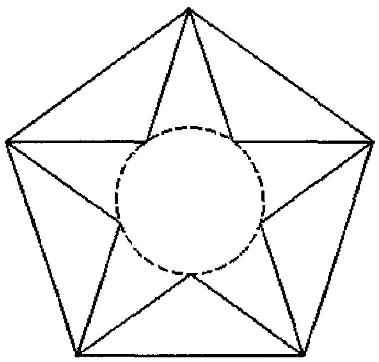

Fig. 2.4

Note: The most fascinating and comprehensive analysis of all sort of symmetries, can be found in the book written by the mathematician D. Schattschneider ("Visions of Symmetry") who has devoted much of her research to studying the work of the famous graphic M. C. Escher. In this beautifully illustrated volume, she presents a detailed symmetry inventory of Escher's periodic drawings with one motif, two or more motifs and non-interlocking patterns.

3. Symmetries in Architecture

An interesting application of the above mentioned point group symmetry of the square, has been recently presented by Jin-Ho Park. Two housing plans have been analyzed by him to show how the mathematical methods of symmetry operations are employed as thematic elements in the unit design as well as the variations in the planning of the site.

The first example was Frank Lloyd Wright's social housing project, called the Quadruple Building Block. Wright (1867-1959) used a standard unit plan for the project, based on his earlier design of his own Home & Studio, Oak Park, Illinois, 1889. A version of this project was originally published as a vignette in the 'Ladies Home Journal' article of 1901 on his "Small House with Lots of Rooms in It". Whereas the unit itself was asymmetric, various local symmetries were involved in the unit. The assembly in their site layout included two types: one is the pinwheel type and the other is the mirrored reflection type. Each house was set on four equally subdivided lots, sharing a common backyard in the center for all four houses. The whole complex of the housings was laid out in a way that their elevation could be varied according to their arrangement. The Quadruple Block plan was a scheme with which Wright was actively concerned from 1902 till the next 10 or 15 years. We cannot be sure that his purpose was a practical one at all because they are not inexpensive row houses designed for mass production; they are substantial dwellings on half-acre plots. Unfortunately, he never found either four families who wanted identical dwellings or a real state investor who could believe that such blocks would constitute a salable commodity. But he never abandoned this revolutionary idea of urban design. Wright's next housing designs share a common compositional theme: the pinwheel type of symmetry, characteristic of the point group symmetry of the square (Figs. 3.1 and 3.2).

Fig. 3.1

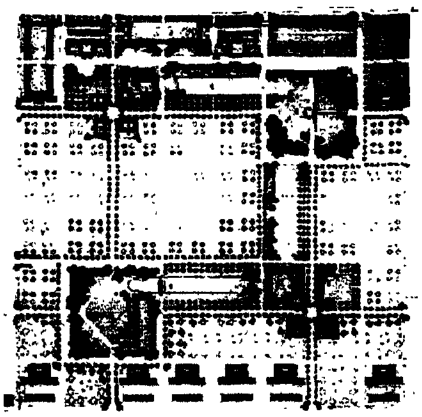

Fig. 3.2

The second example was Rudolf Michael Schindler(1887-1953) unexecuted Shelter project, developed from 1933 to 1942. In this project, the overall floor plan was based on a 5-foot square grid. Along with the square grid, the symmetry governed the internal structure of the spatial composition in each shelter unit as well as the unit variations. The internal organization was subdivided by the removable closet partitions for spatial flexibility and set along the pinwheel type of rotational symmetry. Then the garage was added to any side of the house as a separate unit. But instead of providing only a standard unit with fixed layouts, a sequence of variations were provided, variations that shared the same underlying organization system. Using the symmetry operations of the square, new eight different shelter designs were produced. Taking advantage of the possibilities of combining units into groups, Jin-Ho Park presented an experimental grouping exercise using the shelter unit corresponding to the eight transformations of the symmetry of the square in an abstract level, quite similar to what Wright envisioned in his Quadruple project.

4. Forbidden symmetries

Quasicrystals, which belong to a class of quasiperiodic systems, gave diffraction patterns that show local n-fold rotational symmetry forbidden in Crystallography: n = 5, 8, 10, 12. In this context, quasilattices can be thought as mathematical discrete sets supporting Bragg peaks or atomic sites. They play the same role as lattices do for crystals.

Quite recently, discrete sets of numbers, the β-integers $\mathbf{Z}_\beta$, have been proposed as numbering tools for coordinating quasicrystalline nodes in 1, 2 or 3 dimensions, and also the Bragg peaks in diffraction patterns. In the observed cases:

Golden Mean (penta- or decagonal quasilattices):

$$\beta = \phi = \frac{1+\sqrt{5}}{2} = 2\cos\frac{\pi}{5}$$

Silver Mean (octagonal quasilattices)

$$\beta = \sigma_{Ag} = 1+\sqrt{2} = 1+2\cos\frac{\pi}{4}$$

Subtle Mean (dodecagonal quasilattices)

$$\beta = \varphi^3 = 2+\sqrt{3} = 2+2\cos\frac{\pi}{12}$$

An important mathematical common characteristic is that the continued fraction expansions of these irrational numbers are the following:

$$\phi = [\overline{1}];\ \sigma_{Ag} = [\overline{2}];\ \phi^3 = [\overline{4}]$$

where the usual notation [...] for continued fractions is used. All of them are purely periodic continued fraction expansions.

The relevant scale factor β is a quadratic Pisot-Vijayaraghavan number or more simply, a PV number, i.e. an algebraic integer $\beta > 1$ which is solution to equations of the type

$$x^2 = ax \pm 1 \qquad (a \in \{1, 2, 4\})$$

such that all respective second roots β' (called Galois conjugates of β) have modulus strictly smaller than 1.

These PV numbers are a subset of the Metallic Means Family (MMF), introduced by the author. The more outstanding member of this family is the well known Golden Mean, then comes the Silver Mean, the Bronze Mean, the Copper Mean, the Nickel Mean and many others. These positive quadratic irrational numbers intervene in the determination of the quasi-periodical behavior of non-linear dynamical systems, being therefore an invaluable key in the search of universal roads to chaos. Besides, the members of the MMF satisfy simultaneously many additive and geometric properties, having been in consequence advantageously adopted as bases of many architectonic systems of proportions.

The role played in lattice theory by the ring of integers **Z** and the planar rotational compatibility condition $\rho = 2\cos(2\pi/n) \in \mathbf{Z}$ is in quasilattices replaced by $\rho \in \mathbf{Z}_\beta$.

In terms of the members of the MMF, quadratic Pisot numbers have the following equivalences

n	ρ	β	β '
5	$2\cos 2\pi/5 = 1/\phi$	$1+\cos 2\pi/5 = \phi$	$1-\phi$
8	$2\cos 2\pi/8 = \sigma_{Ag}-1$	$1+2\cos 2\pi/8 = \sigma_{Ag}$	$1-\sqrt{2}=2-\sigma_{A}$
12	$2\cos 2\pi/12 = \sqrt{3}$	$1+2\cos 2\pi/12 = 1+\sqrt{3} = [2,\overline{1,2}]$	$1-\sqrt{3} = [\overline{1,2}]$
12	$2\cos 2\pi/12 = \sqrt{3}$	$2+2\cos 2\pi/12 = 2+\sqrt{3} = [3,\overline{1,2}]$	$2-\sqrt{3} = [1,\overline{1,2}]$

The discovery of quasi-crystals with crystallographically forbidden symmetries is one of the most striking examples where a pure symmetry analysis determines mathematically forbidden symmetries appearing in a new solid state of matter.

Interesting to mention, Silver films with a close-packed structure modulated by a "Silver Mean" quasi-periodic sequence have been experimentally obtained. This represents a new type of quasicrystal that is fundamentally different from the commonly known quasicrystals, which possess at least two-dimensional rotational order.

5. Symmetry and fractals

Fractals are geometric configurations with a built-in self-similarity, that is, configurations that remain invariant in the presence of "*scale changes*". More simply, they possess borders, surfaces or internal structures with patterns that zoomed and zoomed until infinity, are invariant, exactly or statistically. Fractal structures are important to study because they are intrinsically related to the important notion of "*chaos*". A physical process is said to be "*chaotic*" in relation to its dynamics that means, when it is impossible to make any type of prognosis about its future evolution, since it is verified that very similar initial conditions give rise to system behaviors that differ enormously among them. To be able to find the connection between fractal structures and chaotic processes, it is necessary to geometrize the system dynamics. Using computerized graphics, it is feasible to detect fractal patterns that dynamically are considered as "*universal scenarios of the roads to chaos*".

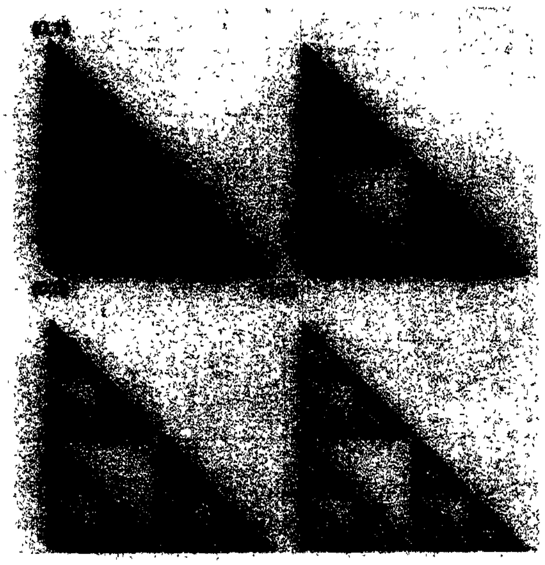

Fig. 5.1

Let us consider some variations of a classical geometric fractal: the so-called Sierpinski gasket, depicted in Fig. 5.1. Beginning with a square, it is possible to set three contractions that reduce the initial square by a factor of 1/2 and place the resulting square appropriately

$$v_1(x,y) = \left(\frac{x}{2}, \frac{y}{2}\right)$$
$$v_2(x,y) = \left(\frac{x+1}{2}, \frac{y}{2}\right)$$
$$v_3(x,y) = \left(\frac{x}{2}, \frac{y+1}{2}\right)$$

Returning to the 8 symmetry transformations of a square, we may define a family of patterns specifying a triplet w_1, w_2, w_3, where each w_i is given by the product of transformations $w_i = v_i \; d_k$ for $k = 1,2,...,8$ and $i = 1,2,3$. This makes altogether $8^3 = 512$ different triplet's w_1, w_2, w_3, and each one describing a specific pattern. To illustrate this amazing variety of fractal patterns, all of which are close relatives of the Sierpinski gasket, let us consider 8 different sets of transformations, which are symmetric with respect to the diagonal and the corresponding fractal patterns (see Fig. 5.2).

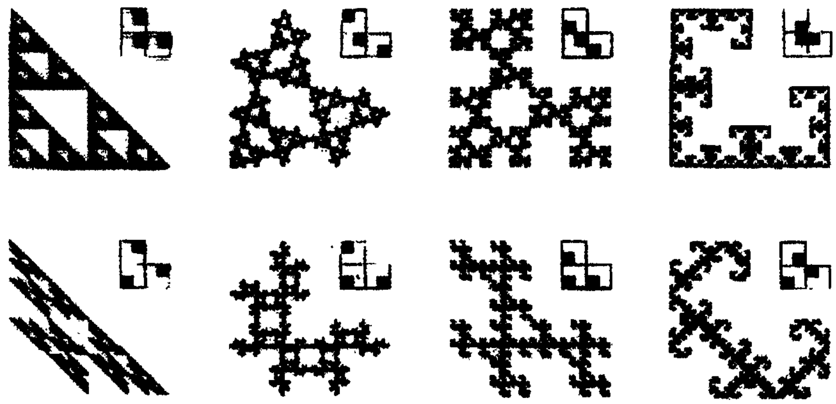

Fig 5.2

This theoretical description is based on the so-called *"Hutchinson operator"*, introduced in 1981 by the mathematician J. Hutchinson. Using this operator, M. F. Barnsley was able in 1985, to devise an IFS (Iterated Function System) with which he succeeded in getting fractal images that were so near from a natural image as desired.

6. Symmetry and visual design

Ljubisa Kocic, a Serbian mathematician, has developed a simple technique for converting real numbers into linear ornaments and vice versa. To achieve this visualization of real numbers, he has recently presented two algorithms A and B, based on the continued fraction expansion of a real number.

Algorithm A

1. Take the continued fraction expansion of a real number $x = [a_0, a_1, a_2, \cdots]$. Cut the sequence $\{a_k\}$ up to the (m + 1)-th term to obtain $a_0, a_1, \cdots, a_m$.
2. Replace a_i by $a_i = a_i (\bmod 8)$. The number x will be represented by $a_0', a_1', \cdots, a_m'$, where $a_i \in \{0,1,\cdots,7\}$ and it is denoted as $\alpha(x) = (a_0', a_1', \cdots, a_m')$..

3. Starting from $(x_0, y_0) = (0,0)$ create a sequence of points $\{(x_k, y_k), k = 1,2,\cdots,m+1\}$ following the rule

$$x_{k+1} = x_k + p$$
$$y_{k+1} = y_k + q$$

where p and q are given in the following table

a_i'	1	2	3	4	5	6	7	0
P	1	1	0	-1	-1	-1	0	1
Q	0	1	1	1	0	-1	-1	-1

4. Draw the polygonal line $\beta(x) = \{(x_0, y_0), (x_1, y_1), \cdots, (x_{m+1}, y_{m+1})\}$.

Note: This table encodes the direction of the path of a moving particle according to the compass rose:

E, NE, N, NW, W, SW, S, SE.

Curiously, this pseudo-random Brownian walk was used by Berthelssen et al. to investigate the quantity of information hidden into the DNA molecule chain.
It is easy to see that Algorithm A, applied to the Golden Mean

$$\phi = \frac{1+\sqrt{5}}{2} = [1,1,\cdots],$$

will produce a horizontal line. The same will happen with all the Metallic Means that have a purely periodic continued fraction expansion, of the form $[\overline{n}]$. But there is a lot of very interesting Metallic Means that originate very nice Brownian friezes, so called because the algorithm produces a kind of Brownian motion trajectory. In fact, since the members of the Metallic Means family are defined as the positive solutions of quadratic equations of the form $x^2 - px - p = 0$ (p, q natural numbers), i.e.

$$\sigma(p,q) = \frac{p+\sqrt{p^2+4q}}{2},$$

it is possible to apply Algorithm A to produce beautiful Brownian friezes, like the shown at Fig. 6.1 that correspond to the following Metallic Means

$$\sigma(3,3)=\frac{3+\sqrt{21}}{2}=1+\frac{1+\sqrt{21}}{2}=1+[2,\overline{1,3}]$$
$$\sigma(1,8)=\frac{1+\sqrt{33}}{2}=[3,\overline{2,1,2,5}]$$
$$\sigma(1,32)=\frac{1+\sqrt{129}}{2}=[6,\overline{5,1,1,2,3,2,1,1,5,11}]$$
$$\sigma(1,578)=\frac{1+\sqrt{2313}}{2}=[24,\overline{1,1,4,1,5,5,5,1,4,1,1,47}]$$

Notice that all these Metallic Means are periodic of the form $[m,\overline{n_1,n_2,\cdots}]$ and these periods are "palindromic" i.e. the periods are symmetric about their centers, except for the last number of the period, which equals $2m-1$.

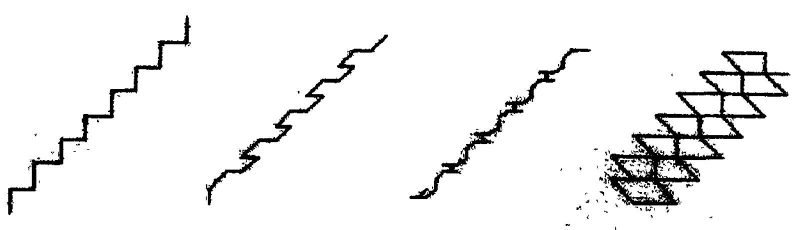

Fig. 6.1

Algorithm B

1. Take the continued fraction expansion of x.. Cut the sequence $\{a_k\}$ up to the (m+1)-th term to get $a_0,a_1,\cdots,a_m$.
2. Create a sequence of pairs $\{a_k,a_{m-k}\}, k=0,1,\cdots,m$.
3. Rotate each point $A_k=(a_k,a_{m-k})$ counterclockwise about the origin and draw this part of the polygonal line $\gamma(x,m)=A_0,A_1,\cdots,A_m$.

The graph $\gamma(x,m)$ is a symmetric figure with respect to the y-axis and is called the "symmetrogram" of x.
The reason of introducing this second algorithm is that the "Brownian frieze" method produces friezes that are not univocal, that is, two different numbers may have the same graph.
Let us apply Algorithm B to irrational numbers containing e = 2.71828..., the basis of natural logarithms.

For example:

$\gamma(e,50)$

$\gamma(e^2,65)$

$\gamma(e^3,30)$

$\gamma(e^{1/4},70)$

produce the symmetrograms of Fig. 6.2.

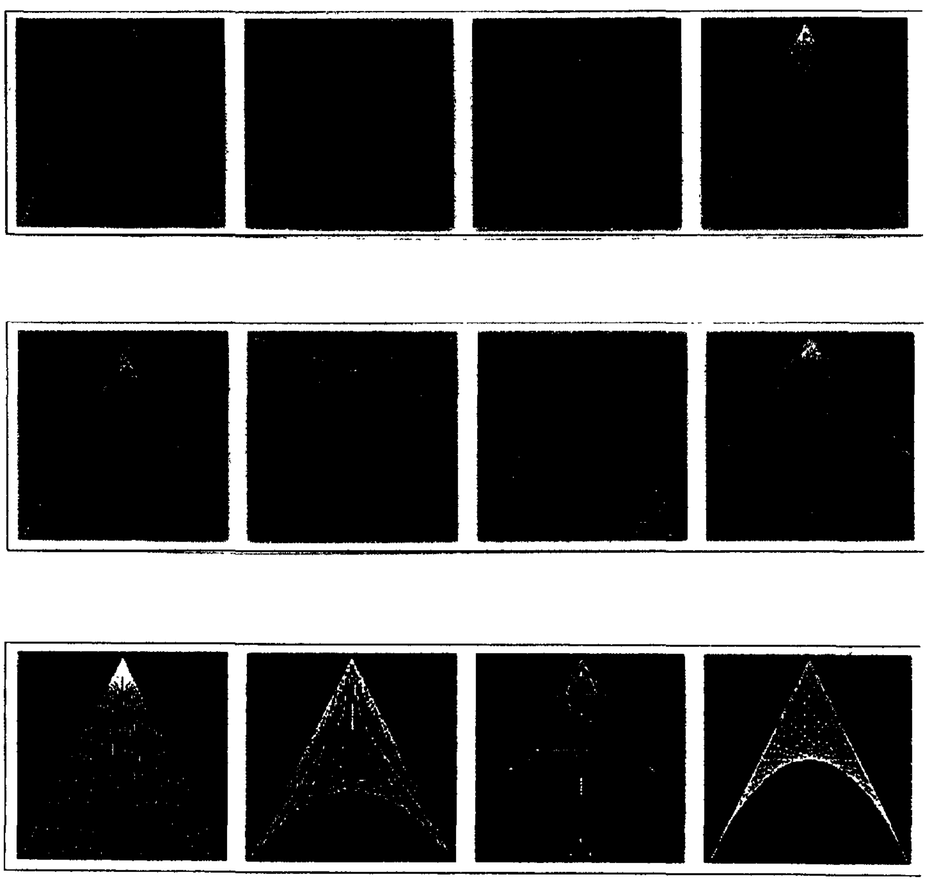

Fig. 6.2

Summarizing, the application of these two algorithms for visualization of real numbers produce patterns and ornaments that range from straight lines over periodic and cyclic to chaotic ones.

7. A visual notation for rational numbers

The Australian mathematician Julie Tolmie has presented at the ISIS-Symmetry Fifth Interdisciplinary Symmetry Congress and Exhibition held at Sydney, Australia, July 8-14, 2001, an interesting methodology to construct a visual notation for rational numbers. Rational numbers *mod* 1 are represented as equivalence classes of pairs (p,q). The torus, with its two directions of rotational symmetry is used as a phase space; its longitudinal cycle for the denominator and its meridian cycle for the numerator. Using cyclic motion and a discrete map of $\{1,2,3,\ldots,37\}$ into the RedGreenBlue RGB-color space, the numerator is encoded as cyclic permutations of color coded dots. These configurations are placed in the appropriate meridian slices (in radial vertical planes), being the result a three dimensional navigable object contained in a torus (a visual abstract is provided at the site *http://www.ozemail.com.au/~jatolmie/isis0.html*).

As is well known, the rational numbers *mod* 1 have a Farey tree, structure, which is called Stern-Brocot tree because it was independently, discovered by the German mathematician Moritz Stern (1858) and the French clockmaker Achille Brocot (1860). To introduce rotational symmetry, the Farey tree is made circular and embedded in the longitudinal plane of the torus. Rational numbers are then defined as curve segments, which span the longitudinal region, bounded by its Farey parents. These curve segments are cut from curves, which wind around the torus at a constant integer velocity and in this way, a three-dimensional navigable object is made.

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