PROPORTIONS AND DISSECTIONS IN POLYGONS

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Fields of interest: Geometry, proportion, symmetry (Art and Architecture, Mathematics with Paper Folding, Phyllotaxis, and Mathematics in Daily Life).

Publications and/or *Exhibitions*:

- Proporciones y Arquitectura. ICME, (Congreso Internacional de Educación Matemática) Sevilla 1996.

- Proportions mathématiques et leurs applications à l' architecture. Bulletin IREM (Institut de recherche pour l'enseignement des mathématiques de Toulouse) Toulouse. France, 1998, 39 pp.

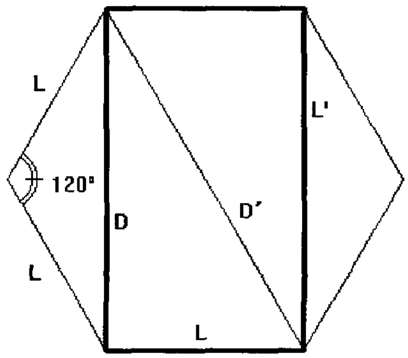
- Papiroflexia y proporciones dinámicas en rectángulos. JAEM (Jornadas Nacionales para la Enseñanza y Aprendizaje de las Matemáticas.) 1999

- Mathematics: a determinant strategic for the development of an architectonic project. Symposium on Symmetry. Alhambra 2000, Granada (Spain)

-Paper folding and proportions in polygons. ISIS Symmetry Congress & Exhibition. Sydney 2001.

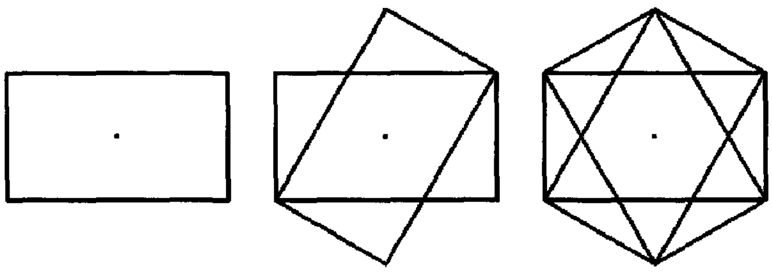
***Abstract**: In this work I will focus mainly on some proportions and dissections in polygons, particularly hexagons and octagons. The classical method of the decomposition of polygons into smaller parts that can be rearranged to form other polygons is a pedagogical tool to understand the geometry of polygons and their properties. An interesting problem to be solved en each case is to find an equivalent polygon by using the minimum number of pieces. I will show some dissections of the hexagon and the octagon with their equivalent figures. In the other hand, teaching mathematics through paper folding is an alternative way to understand them. I will present in this exposition some constructions of the hexagon and the octagon by using these techniques. My interest focuses on researching the underlying mathematics involved in the paper folding process. In each case (hexagonal and octagonal) it will be calculated the relationships of proportion between diagonals and sides the same in convex polygons as in their stars. I will establish also the concept of polygonal proportion (PP) using it to calculate the hexagonal and octagonal global proportion.*

We define the proportion of the regular polygon P_n, written $p(P_n)$, as the quotient between the area $A(P_n)$ of the polygon and the square of the length of its side L: $p(P_n) = \frac{A(P_n)}{L^2}$.

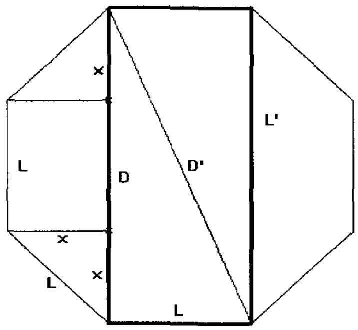


If H_6 is a regular hexagon with side L, this definition gives us $p(H_6) = \frac{A(H_6)}{L^2} = \frac{\frac{3L^2\sqrt{3}}{2}}{L^2} = \frac{3\sqrt{3}}{2}$.

In the hexagon we also have the following proportion: $\frac{D}{L} = \sqrt{3}$. We can construct a hexagon and its star 6/2 by using paper folding. We get a strip of paper with the same proportion than the rectangle $\frac{D}{L} = \sqrt{3}$. Rotate the strip an angle of $\frac{\pi}{3}$ around its center and repeat this operation once more.



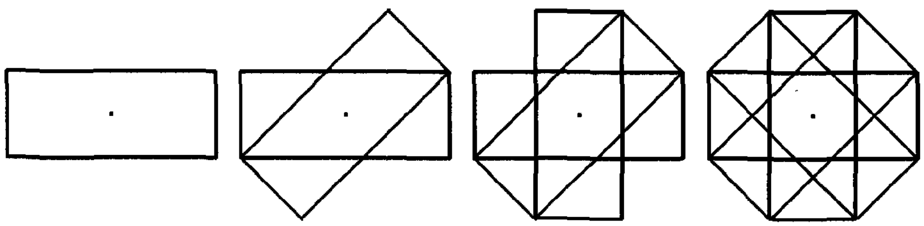
In case of the regular octagon O_8 with side L we have: $p(O_8)=\frac{A(O_8)}{L^2}=2\theta=2(\sqrt{2}+1)$ where θ denotes the silver number.



In the figure, we have $L^2=2x^2;\quad 2x+L=D$, solving this system, we obtain: $x=\frac{L}{\sqrt{2}}$;

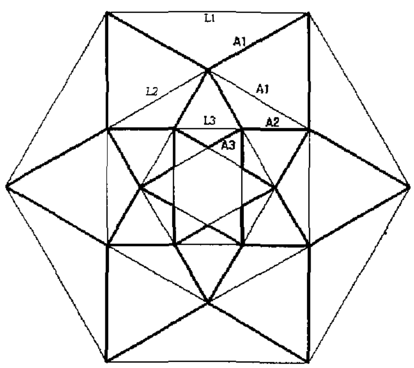
therefore $2\left(\frac{L}{\sqrt{2}}\right)+L=D\Leftrightarrow\frac{D}{L}=\sqrt{2}+1=\theta$.

By rotating the silver rectangle (a rectangle with proportion θ) an angle of $\frac{\pi}{4}$ around its center, we can obtain an octagon and its star 8/3.



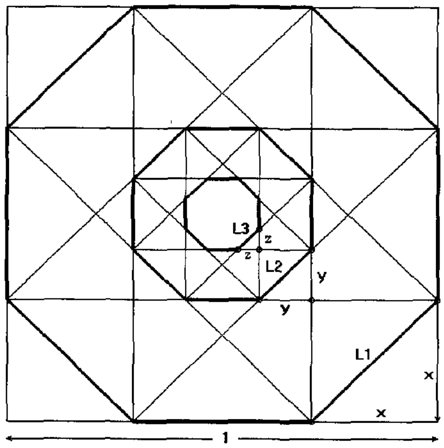
RATIOS OF LENGTH OF SIDES IN HEXAGONS AND HEXAGRAMS 6/2

Let $\{L_i\}$be the decreasing sequence of the length of the sides of the convex hexagons (see figure). This sequence is a geometric series of ratio $\frac{L_{n+1}}{L_n}=\frac{1}{\sqrt{3}}$ and whose sum is



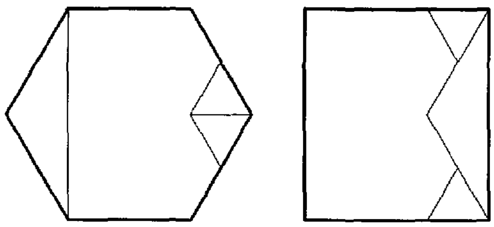
$S=\dfrac{1}{1-\dfrac{1}{\sqrt{3}}}=\dfrac{3+\sqrt{3}}{2}$ If we consider the sequence $\{A_i\}$ of lengths of the hexagrams, we also obtain the same relation. $\dfrac{A_{n+1}}{A_n}=\dfrac{1}{\sqrt{3}}$.

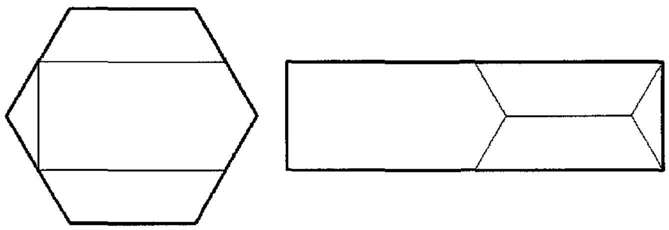
RATIOS OF LENGHTS OF SIDES IN OCTAGONS AND THEIR STARS 8/3



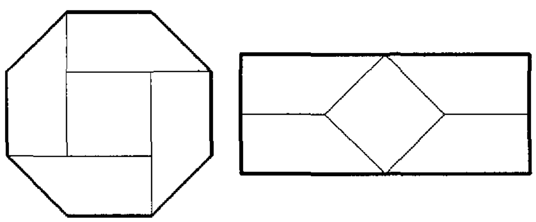
Let $\{L_i\}$ be the decreasing sequence of the length of sides of convex octagons (see figure). This sequence is a geometric series of ratio $\dfrac{L_{n+1}}{L_n}=\dfrac{1}{\theta}$

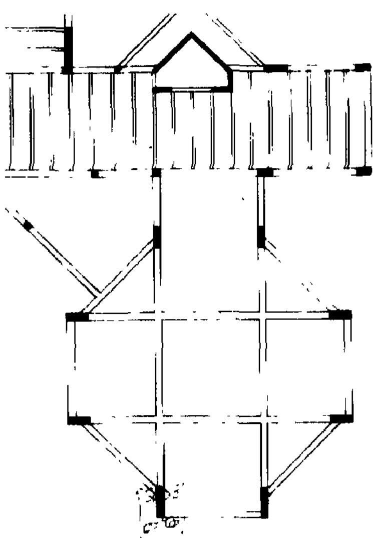
The sum of the series $\{L_i\}$ is: $S=\dfrac{1}{1-\dfrac{1}{\theta}}=\dfrac{\theta}{\theta-1}=1+\dfrac{1}{\theta-1}$. If we consider the decreasing sequence of lengths of sides $\{A_i\}$ in the star polygon **8/3,** we also obtain the same relation. $\dfrac{A_{n+1}}{A_n}=\dfrac{1}{\theta}$.





In the following dissections of the hexagon and the octagon we can rearrange the pieces to form equivalent rectangles.





This interesting dissected octagon appears in CEAC (Center for the Study of Contemporary Art) in Barcelona (Spain); it is conceived by J. L. Sert. This kind of division of the octagon into nine parts is also useful in pavements, because it can be constructed using only two types of tiles: a square and its half. However, such an octagon is irregular.

References

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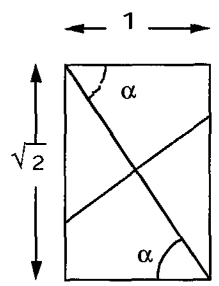
ADDENDUM: QUESTIONS ABOUT THE EXPOSITION

It is possible the construction of the regular pentagon by using a DIN A4 sheet of paper?

The answer was:

A Din A 4 sheet of paper is not suitable for the construction of a regular pentagon because the proportion of a DIN A4 is the irrational number $\sqrt{2}$, and the proportion of the regular pentagon involves the golden number $\frac{1+\sqrt{5}}{2}$.



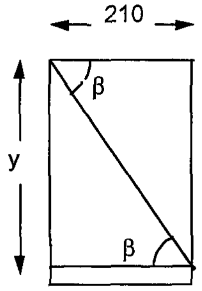
The right construction of the regular pentagon starting with a DIN A 4 sheet of paper requires to cut a small strip ≈ 8 mm (7,96 mm), making the long side of the sheet of paper shorter.

The proof is the following:

The internal angle of a pentagon is $\frac{540^{\circ}}{5} = 108^{\circ}$.

Starting from a DIN A4

$\mathrm{tg}\alpha = \sqrt{2} \Rightarrow \alpha = \mathrm{actg}\sqrt{2} \approx 54{,}74^{\circ}$.



If we fold through the transversal line we obtain the angle $2\alpha \approx$ 109,47 °, that which is different from 108°.

We need another rectangle with $\beta = \frac{108}{2} = 54^{\circ}$, $\mathrm{tg}54^{\circ} = 1{,}376$, therefore if we denote by "y" the long side of the rectangle A4, $\mathrm{tg}\beta = \frac{y}{210} \Rightarrow y = 210.\ \mathrm{tg}54^{\circ} = 289{,}04$ mm.

And since the length of the long side of a DIN A4 is 297 mm, we must cut: 297 - 289,04 =7,96 mm. ≈ 8mm.

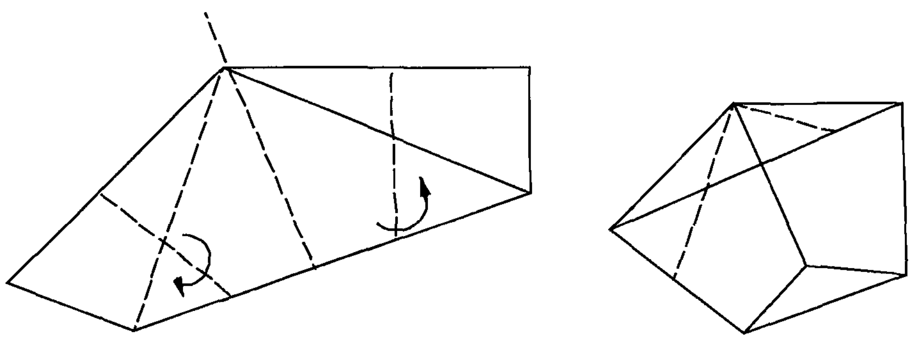
Therefore,

-Cut 8 mm to the long side of a DIN A4 format. (Then, cut along the short side, see figure)

-Join the two opposite vertices of paper and fold it.

-Fold the figure along its axes of symmetry

-Fold the smallest sides once more until they coincide with the axis of symmetry.



THE REGULAR PENTAGON IS OBTAINED

Another question was: What about the heptagon? Has it been constructed?
The answer was:

So far no heptagon construction using paper folding has been executed. The only regular polygons whose construction was studied by the author were those that can be drawn by ruler and compasses.