TESSELATIONS OF EUCLIDEAN, RIEMANNIAN AND HYPERBOLIC PLANE

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Abstract: *The very first human perception of this world was plane. Then one sphere among others in the universe. Or is it just a small part of the hyperbolic space? Different features of Euclidean, Elliptic and Hyperbolic geometry are presented through all types of planar edge-to-edge tessellations with regular polygons as tiles (not all congruent) and vertices of the same type, including regular, uniform, non-uniform and coloured tessellations. The most intriguing, hyperbolic geometry is the main topic of the algorithm implemented in Mathematica 4.0® that provides tools for creating and drawing tessellations and their animation under different isometries, in both Poincare models and Klein disk model. The package provides additional data such as: geometry and type of tessellation, number of all possible realizations, angles of tiles and transformation rules between vertices upon which the tessellation is constructed.*

1 INTRODUCTION

According to the Oxford Dictionary the word tessellate means to form or arrange small squares in a checkered or mosaic pattern. Being familiar or not, with this or any other (more) formal definition, everyone can recognize tilings like brick walls and tile floors.

Tessellation as an art predates human history- Platonic polyhedra were well-known in antiquity and a toy regular dodecahedron was found in Padua in Etruscan ruins dating from 500 B C. Excellent examples of tessellations can be found in Moorish architecture in Spain and Islamic architecture in the Middle East; not to mention Japanese and Chinese designs. Artists such as Albrecht Durer and Pierro della Francesca made drawings of many of the semi-regular polyhedra. Johannes Kepler was the first to give a complete description of such figures in his Harmonices Mundi which appeared in 1619. Except this initial study, some formal mathematical investigation took place before the end of 20th century. We must mention Grunbaum, Shephard, Robin, Sommerville, Andreini, M S. Escher and that much was done also by chemist and crystallographers .

2 DEFINITIONS

Def. 1. A ***plane tiling*** is a countable family of polygons P_1, P_2,... called tiles such that:

1. Their union is the entire plane
2. Interiors of the tiles are pairwise disjoint

Def. 2. An ***edge-to-edge tiling*** is plane tiling where each two tiles intersect along a common edge, common vertex or not at all

Def. 3. A vertex of a tiling is said to be of type $n_1, n_2, \dots, n_N$ or that $(n_1, n_2, \dots, n_N)$ is the ***vertex type*** (configuration) if the polygons about this vertex in cyclic order are an n_1-gon, an n_2-gon, and an n_N-gon where N is the number of polygons at each vertex.

Def. 4. An edge-to-edge tiling is called ***Archimedean*** if

1. Each tile is a regular polygon (not all congruent)
2. All vertices are of the same type.

Def. 5. A ***regular tiling*** is an edge-to-edge tiling whose tiles are congruent to a single regular polygon. The other name is monohedral tiling with regular polygons as tiles.

Def. 6. A *semi-regular* or ***uniform tiling*** is an Archimedean tiling whose symmetries are transitive on its vertices.

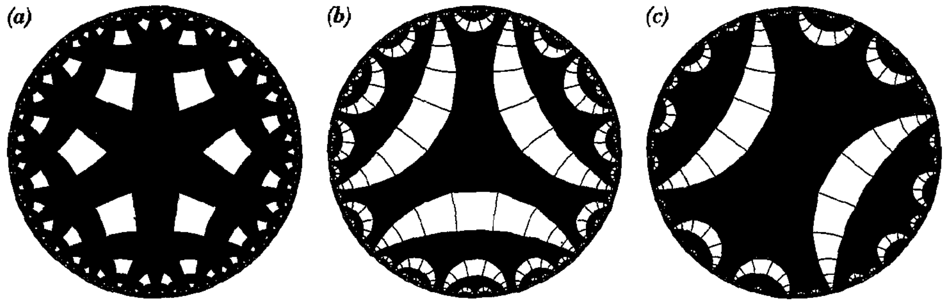

Figure 1. (4,4,4,6): two uniform (a, b) and non-uniform realization (c): colours make distinction between polygons that can not be mapped to each other by global symmetries

The following array shows the relations between different types of tessellations:

Regular $\subset$ Uniform $\subset$ Archimedean $\subset$ Edge-to-edge $\subset$ Plane Tessellations

For the rest of this paper we restrict our attention to the tessellations whose tiles are regular polygons and vertices are of the same type. This gives us an opportunity to define tessellations with their vertex configuration, although this correspondence is not always one-to-one. The vertex symbol uniquely determines the type of a tiling in Euclidean and Elliptic plane (except (3,4,4,4) which has two realizations: rhombicuboctahedron and a non-uniform polyhedra, Fig. 3). However, non-congruent tessellations in the hyperbolic plane may have the same vertex symbol (Figs 1 and 2).

3 THEORY AND ALGORITHM

The general solution for constructing tessellations is based on a few rather simple ideas that we shall briefly explain. The algorithm can be divided in two logical parts:

Step 1. Vertex configuration $\rightarrow$ transformation rules for neighbouring vertices

Step 2. Transformation rules for neighbouring vertices $\rightarrow$ graph $\rightarrow$ tessellation

Step 1. The starting point of the algorithm and the only input is the vertex configuration. In order to find all possible realizations for a given vertex configuration we developed a very general algorithm Our initial assumption is that each polygon belongs to a different equivalence class (each presented with different colours in our figures) under the symmetries of the whole tessellation (global symmetries) and then we, if necessary, decrease the number of classes. In each iteration we look for transformation rules i.e. mappings of the corresponding vertices and check if they really do form a tessellation.

In this way we are able to find both uniform and non-uniform tessellations i.e. all Archimedean tessellations including enantiomorphic realizations (Fig 4) and all coloured realizations (Fig. 2.a, 2.b).

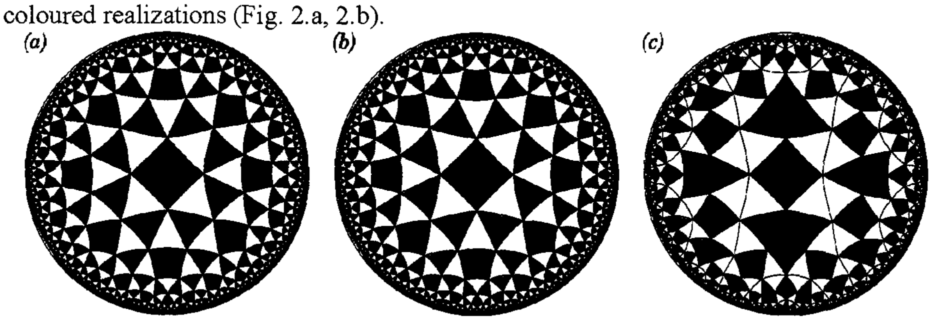

Figure 2. (3,3,3,3,3,4): uniform realization (a) and its coloured version (b), and one of the non-uniform realizations (c).

Step 2. Obtained transformation rules posses two important properties: they are uniquely defined and sufficient for creation of the whole tessellation starting from any vertex. So, once we have transformation rules we construct the graph and tessellation at the same time, using following ideas.

♦ Vertex configuration imposes geometry onto the graph. Hence, the graph can be realized only on a 2-dimensional, simply connected, unbounded, complete Riemannian surface of constant curvature, i.e. Euclidean, Elliptic or Hyperbolic plane [2].

♦ In constructing tessellations problems arise on local and global level [5]. On the local level we consider only those images of the original tile obtained by a sequence of symmetries that move it in a way that the tile and its image are not totally disjoint So the local tessellation problem is the question if these adjacent tiles cover a certain area completely enclosing the original one without gaps and overlapping. Global tessellation problem is the same question involving the whole 2-dimensional space. The main point is that in order to determine if the tessellation exists or not, we need to solve only the local problem

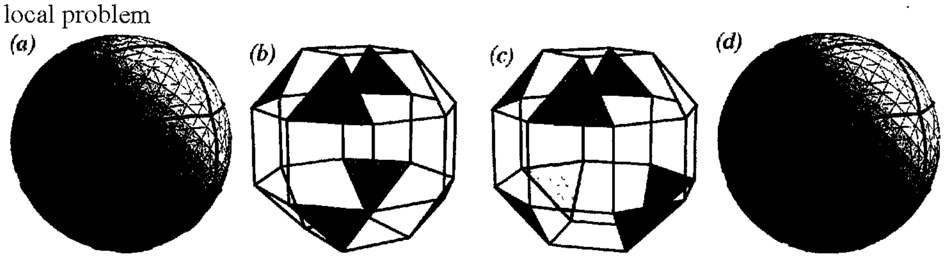

Figure 3. (3,4,4,4): uniform tessellation (a, b) and non-uniform (c, d).

This is the very core of the algorithm and additional calculations are necessary for drawing tessellations [1, 3, 7]. The algorithm contains methods for drawing Euclidean and Elliptic tessellations (wired and coloured model or a corresponding polyhedra, Figs 3, 4c) and for hyperbolic tessellations Mathematica® [6, 8] package L2Primitives [4] is used (both Poincare's and Klein disk model) Furthermore, it is possible to apply various isometries and obtain interesting animations. As a side effect, algorithm provides additional useful information such as geometry of tiling, type of tessellation, number of all possible realizations, angles of tiles [3] and complete listing of transformation rules. The detailed description of the algorithm and the theory behind will be published elsewhere.

4 GALLERY

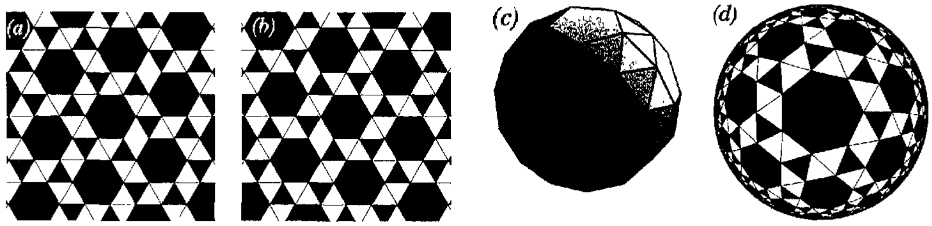

Figure 4. Uniform enantiomorphic tessellations: (3,3,3,3,6) Euclidean (a, b), (3,3,3,3,5) Riemannian (c), and (3,3,3,3,7) hyperbolic in Klein disk model (d).

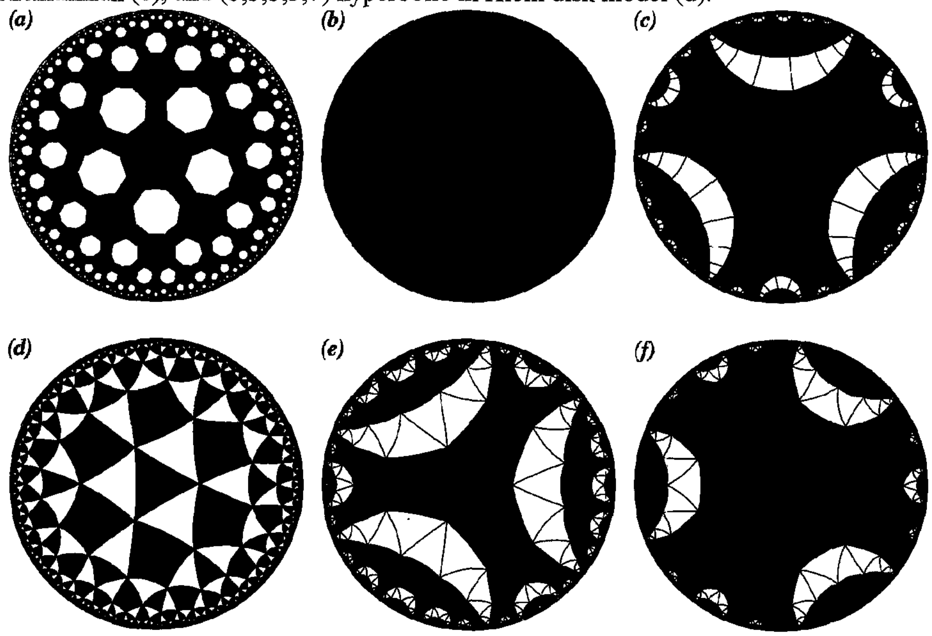

Figure 5. Tessellations with a single realization: uniform (4,8,10) and (3,6,4,6) (a, b) and non-uniform (3,4,3,4,4) (c). Tessellation with multiple realizations (3,3,3,4,3,4): two uniform (d, e) and a non-uniform (f).

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