VISUALIZATION VS. VERBALIZATION, INSIGHT INTO THE MORPHOLOGY OF POLYHEDRA

IRIT WERTHEIM AND NITSA MOVSHOVITZ-HADAR

Name: Irit Wertheim.

Address: Technion – Israel Institute of Technology, Department of Education in Science and technology, Haifa 32000, ISRAEL.

E-mail: weririt@techunix.technion.ac.il.

Fields of interest: morphological approach to 3-D geometry.

Name: Nitsa Movshovitz-Hadar.

Address: Nitsa Movshovitz-Hadar, Ph.D., Head of Kesher Cham - National Center for Mathematics Education, Director of the Israel National Museum of Science, Planning, and Technology, Technion 32000, ISRAEL.

E-mail: nitsa@techunix.technion.ac.il.

Abstract: *The impact of visual representations on understanding, and even more so, on actively doing mathematics, has been intensively researched and is widely recognized. This is particularly true for the study of 3-d geometry. What about the role of verbal descriptions? Are they necessarily needed? Simply redundant? Or are they inappropriate, may be even disturbing? In this paper we attempt at demonstrating the crucial role verbalization plays as a complementary mode to visualization, for dealing with 3d geometry tasks, understandably and insightfully. Our claim is that neither visualization alone nor verbalization in itself suffice for meaningful conceptualisation. The cognitive processes involved will be demonstrated in the context of polyhedra.1*

Introduction - One picture is worth a thousand words, or is it?

In their struggle towards obtaining a sense of meaning for the hypercube (a 4-d cube), Davis & Hersh describe vividly, the important role played by manipulating a visual representation:

"...I was impressed by ... the sheer visual pleasure of watching it. But I was disappointed; I didn't gain any intuitive feeling for the hypercube... I tried turning the

hypercube around, moving it away, bringing it up close, turning it around another way. Suddenly I could feel it! The hypercube had leaped into palpable reality, as I learned how to manipulate it..." (Davis & Hersh, 1981).

"One picture is worth a thousand words". Underlying this well-known saying is the widespread experience that a relatively simple visual representation can replace a lot of, and save hours of talking.

The impact of visual representations on understanding, and even more so, on actively doing mathematics, has been intensively researched in the past decades and is widely recognized nowadays. This is particularly true for the study of 3-d geometry (e.g., Parzysz, 1999; Barwise & Etchemendy, 1991; Zimmermann & Cunningham, 1991). Actually, the employment of visual representations in the study of spatial geometry comes very natural and handy. It is commonly agreed that 2-d drawings and 3-d concrete models are necessary tools, which provide comprehensive understanding of 3-d configurations. What about the role of verbal descriptions? Are they necessarily needed? Simply redundant? Or are they inappropriate, may be even disturbing???

In this paper we attempt at demonstrating the crucial role verbalization plays as a complementary mode to visualization, for dealing with 3d geometry tasks, understandably and insightfully. Neither visualization alone nor verbalization in itself suffices for meaningful conceptualisation. This claim is supported by Kosslyn (1980) and by Stigler (1984) who suggested that visual and verbal representations of the same information have distinct attributes augmenting one another.

To elaborate on the issue we focus our attention to the elementary processes of watching a solid or a 3-d configuration, followed by articulating its structure and properties. We'll show that the latter, promotes insight and understanding, which might not be gained through sheer observation nor by manipulating the visual representations. This we do in the context of convex Deltahedra - - Polyhedra with equilateral triangular faces.

To conclude this introduction, here is a personal anecdote by the first author:

The three Platonic polyhedra made out of equilateral triangular faces, were old "acquaintances" when I was first introduced to the set of convex Deltahedra (Note that if we do not restrict ourselves to convex deltahedra, there are infinitely many deltahedra, e. g., the stellated dodecahedron). The deltahedral dipyramids were quite easy to grasp (a dipyramid is a solid made of two congruent pyramids with a common base). However, there were three other deltahedra, D_{12}, D_{14} and D_{16}, which remained altogether "strangers". It was difficult for me to gain any intuitive feel for them - I was confused. I got a hold of them, only as I was able to construct a verbal description for each (!). "Suddenly I could feel them! They had leaped into palpable reality", just like Davis' experience with manipulating the visual representation of the hypercube.

2 How many convex deltahedra possibly exist?

An n-face deltahedron is a polyhedron with n congruent equilateral triangular faces, conventionally denoted D_n. The name comes from the Greek capital letter delta, which has a triangular shape. An interesting problem is to find how many such convex polyhedra there are.

In trying to exhaust the set of convex deltahedra, we note, first of all, that this set includes three of the five regular/Platonic solids, D_4 - the tetrahedron; D_8 - the octahedron; D_{20} - the icosahedron.

Other convex deltahedra might be obtainable by attaching several triangles side to side in different spatial configurations. Our question is: How many convex deltahedra can thus be obtained?

2.1 How many faces can convex deltahedra possess?

To answer this question we first observe that in each vertex 3, 4, or 5 triangular faces only, can meet. These numbers define the valence of a vertex. Clearly, the valence of all vertices in any given deltahedron is not necessarily the same. Denote V_3, V_4, V_5 - the number of vertices of valence 3, 4, 5 respectively.

Next, note that

(i) The minimum number of faces is four, as in the tetrahedron.

(ii) Since no vertex can have more than five equilateral triangles meeting in it (as six give a flat 3600 angle sum around the vertex), the largest possible convex Deltahedron is a solid having 5 equilateral triangles meet in each of its vertices, i.e., the icosahedron. Thus twenty is the maximum number of faces for any Deltahedron.

Therefore, The total number of faces of any existing convex Deltahedron is in the range between 4 and 20.

2.2 Candidates for existing deltahedra.

First observe that the number of faces in any Deltahedron must be even. To establish it note that each face has three edges and each edge is shared by two faces. Using Euler's formula for simple polyhedra, F(aces)+V(ertices)=E(dges)+2 , we get F=2(V-2) which implies the evenness of the total number of faces. Thus, we conclude that there are only 9 candidates for deltahedra – those having 4, 6, 8, 10, 12, 14, 16, 18 and 20 (triangular) faces.

To determine the various possible deltahedra, we employ Descartes's formula for the total angular deficit of a convex polyhedron. First observe that:

(i) The angular deficit of a vertex of valence 3 is 180°;

(ii) The angular deficit of a vertex of valence 4 is 120°;

(iii) The angular deficit of a vertex of valence 5 is 60°.

Descartes's formula for the total angular deficit of a convex polyhedron yields

$$180°V_3 + 120°V_4 + 60°V_5 = 720°,$$

or

$$3V_3 + 2V_4 + V_5 = 12.$$

Every Deltahedron must satisfy this (Diophantine) equation. This is a necessary but yet insufficient condition for the existence of a Deltahedron.

2.3 The solution

Every solution of the above mentioned (Diophantine) equation, which takes into account the evenness of the total number of faces, is a candidate for an existing Deltahedron. Altogether, there are nineteen possible solutions to the equation. They are listed in Table 1 (Lichtenberg, 1988). However, the only solutions that correspond to actually existing deltahedra are 1, 3-7, 16, and 19, total of eight convex deltahedra (a deltahedra is a polyhedra with equilateral triangular faces).

3. Visual and verbal description of the eight deltahedra.

As we have seen, altogether, there are exactly eight convex deltahedra. This set includes:

- The Platonic solids: The tetrahedron D_4, the octahedron D_8, and the icosahedron D_{20};
- Two dipyramids - D_6, and D_{10}. (D_8 which is also a dipyramid is not included here because it is mentioned above).

  D_6 - triangular dipyramid - two attached tetrahedral.

  (D_8 - square dipyramid - two attached square pyramids, actually the Octahedron).

  D_{10} - pentagonal dipyramid - two attached pentagonal pyramids.

- And - three rather more complicated deltahedra:

  D_{12} - snub disphenoid. This solid can be thought of as a tetrahedron split into two wedges each made of two adjacent faces, which are then joined together by a band of eight triangles.

  D_{14} - triaugmented triangular prism - three square pyramids, each attached to a square side face of a triangular prism.

  D_{16} - gyro elongated square dipyramid - two square pyramids attached to a square antiprism.

Note that an *antiprism* has two congruent regular n-side polygons, as bases, in parallel planes. One base is twisted so that each of its vertices is midway between two vertices of the other, to each of which it is joined. The side faces are triangles. While twisting one of the prism's bases, the square side faces of the prism are folded into triangles, and the result is an antiprism.

3.1 D_{14} and D_{16} – Can you tell the difference?

It is quite difficult to distinguish D_{14} and D_{16} from each other by their visual images. Holding their concrete real models and looking at them from various angles does not help much either. On the other hand verbal descriptions make it quite clear:

D_{14} is a triangular prism with a square pyramid attached to each of its side faces;

D_{16} is a square antiprism with a square pyramid attached to each of its bases.

3.2 “Seeing” and believing

We observed high school and college students attempting to describe verbally these two solids, after they saw their actual models and various drawings. The evidence collected suggest quite clearly, that only after one is able to give a coherent verbal description, one reaches an understanding of the structure of each solid, and is able to “see” the differences between the two solids. - - As if these solids leaped into palpable reality, in Davis’s (ibid) language.

4. A Brief Discussion

Each of us tends to go through an initial "grasping" in which we understand key concepts but cannot converse about them fluently. As our exposure to the material increases, we are able to shape our comprehension through questions, tentative verbalizations, informal talks with others, reorganization of notes, and so forth. Through language, then, we gradually extend the meaning and gain understanding. We not only recognize the structure of the subject, but also verbally manipulate its ideas, expressing its orderliness in personalized and unique ways (Suhor, 1984). Mathematics educators have been busy studying mathematical discourse in the past two decades. It is commonly agreed within this community of researchers that intuition and operational experimentation serve successfully the development of higher level understanding, however, reaching this level necessitate reflective thinking which is anchored in verbal discourse. (E.g., Skemp 1973, Hiebert, and Carpenter 1992, Sfard 1994 and others)

There is a tendency to communicate about visible entities by visual representations, and about formal mathematics notions, by symbols and verbal arguments. Although 3-d geometrical entities are real and visible, the examples given above illustrate that in order to develop a sense of meaning for their structures, their properties and the inter-relationships among them, it is necessary to combine language and pictorial representation. "... thought unexpressed remains immature and eventually dies out.

Language therefore, is not just an expression of otherwise independent and fully formed thought, but rather is a necessary form of the thought's realization." (Vygotsky, 1986). While the visual representation provides an intuitive infrastructure, it is the language that guarantees the transition from the intuitive perception to an analytic-synthetic understanding, and even to innovative findings. For example, while describing in words the process of truncating (cutting off) the vertices of a tetrahedron through mid-edges, students observed that truncating the "top" vertex first, yielded a triangular face parallel to the "basis" of the original tetrahedron. This observation led to the insight that by truncating the other three vertices, a triangular antiprism is obtained. A second thought brought them to the conclusion that the resulted 8 triangular face polyhedron is actually an octahedron, and from here the way was short to the insightful finding that the triangular antiprism and the octahedron are nothing but two sides of the same coin. The property of the octahedron as an antiprism emerged as a result of the discourse among the group of students who had a tetrahedron in front of them and were challenged to study the results of various ways of truncating it. The students arrived at their discovery, totally on their own based upon the combination of the mental image they were able to operate on, and their linguistic ability to follow their imaginary action of truncating and connect it to a recently acquired concept of the antiprism.

Visual representations and mental (pictorial) images, being wholesome and compact supported students' overall structural conception, that can be grasped at one glance, while verbal encoding enabled a process of decomposition of the whole into its parts, analysing the inter-relationship among the parts and the complete new structure. Language played an important role in coming to grips with spatial relationships and structural properties. Using 'literary' forms in making connections, helped students in making sense of mathematical constructs, and in remembering it. Language is the medium within which the creation of new concepts takes place. We are not questioning the power of visualization. As Sfard (2000) noted: "Availability of visually manipulability means, either actual or only imagined, underlies our ability to communicate on the objects and operate them discoursively". On the other hand, we suggest that verbal representations for visual entities are crucial for the understanding of those entities. The concepts we are dealing with belong to the domain of concrete objects or processes, usually, accessible to us through perceptual experience. As these concepts are more complicated, the role of language and verbalization becomes more crucial.

Students come to recognize the properties of a solid, by describing them verbally. While verbalizing, students organized their visual impressions by comparisons: distinguishing which aspects of the shape are to be noticed and which ones are to be ignored, which properties are similar to already known objects, and which are different. The immediate implication is that whoever strives to become a professional, must develop the ability to express verbally, visual observations and to visualize verbal ideas mentally.

Describing an object verbally is a process that involves the decomposition of the whole, followed by sequential reassembling of the parts. As early as 1969, Paivio observed that visual imagery has many of the properties of a spatially parallel system, whereas verbal processes are better suited for handling sequential, serial information (Paivio 1969).

It takes cognitive processing to make sense of visual information, be it a 2d representation or a concrete 3-d model. The interplay between visualization and verbalization is the key to cognitive processing of that sort. The enhancement of this combination is therefore at the heart of a literate approach to any profession that has to do with spatial relations - - Architecture, Engineering, Mathematics and Arts.

Finally, to balance the claim made so far, we leave you with a dilemma we started this paper with: Is verbal description always appropriate? Is it necessarily needed with no exception? Or is it sometimes simply redundant? Possibly disturbing???

References

Barwise, J. & Etchemendy, J. (1991). Visual Information and Valid Reasoning. In (Ed) Zimmermann, W. & Cunningham, S. Visualization in Teaching and Learning Mathematics. Mathematics of America. USA

Davis, P.J. & Hersh, R. (1981). The Mathematical Experience. Birkhauser Boston. U.S.A.

Hiebert, J., & Carpenter, T. P. (1992). Learning and teaching with understanding. In D. A. Grouws (Ed.), Handbook of research on mathematics teaching and learning (pp. 65–97). New York: Macmillan.

Kosslyn, S.M. (1980) Image and Mind. Harvard University Press.

Lichtenberg, D.R. (1988). Pyramids, Antiprisms and Deltahedra. Mathematics Teacher. Vol. 81 Num. 1 pp. 261-265.

Paivio, A. (1969). Mental imagery in associative learning and memory. Psychology Rev. 76 pp. 241-3.

Parzysz, B. (1999). Visualization and Modelling in Problem Solving: From Algebra to Geometry and Back. Proceedings of the 23rd Conference of the PME.

Sfard, A. (2000). Steering (Dis)Course Between Metaphors and Rigor: Using Focal Analysis to Investigate an Emergence of Mathematical Objects.

Sfard, A. (1994). Reification as a birth of a metaphor. For the Learning of Mathematics, 14(1), 44-55.

Skemp, R.R. (1973). Relational Understanding and Instrumental Understanding. Mathematics Teaching.

Stigler, J. (1984). Mental Abacus: The effect of abacus training on Chinese children's mental arithmetic. Cognitive Psychology 16 145-176.

Suhor, Charles (1984). Thinking Skills in English and across the Curriculum. ERIC Digest. Eric Product (071); Eric Digests (selected) (073).

Vygotsky, L. (1986). Thought and Language. The MIT Press Cambridge. Massachusetts.

Zimmermann, W. & Cunningham, S. (Eds). (1991). Editors' Introduction: What is Mathematical Visualization. In: Visualization in Teaching and Learning Mathematics. Mathematics of America. USA