

ARTISTIC PATTERNS WITH HYPERBOLIC SYMMETRY

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Introduction

Probably the first repeating patterns of the hyperbolic plane were triangle tessellations (see Figure 1 below) which, though attractive, were not originally created for artistic purposes. Almost certainly the Dutch artist M. C. Escher was the first person to combine hyperbolic geometry and art in his four patterns *Circle Limit I*, *Circle Limit II*, *Circle Limit III*, and *Circle Limit IV* — see Catalog Numbers 429, 432, 434 (and p. 97), and 436 (and p. 98) of [Locher, 1982]. It is exacting and time-consuming to create such patterns by hand as Escher did. In the late 1970's, the power of computers was applied to the problem of creating such patterns. Since then, much progress has been made in this area which spans mathematics, art, and computer science [Dunham, 1986a], and [Dunham, 1986b].

We will begin with a review of hyperbolic geometry, repeating patterns and tessellations, symmetries of hyperbolic patterns, and color symmetry. Then the theory of repeating hyperbolic patterns will be related to that of Euclidean and spherical patterns. Finally, a computer-aided hyperbolic pattern-generation process will be described.

Hyperbolic Geometry

By definition, (plane) hyperbolic geometry satisfies the negation of the Euclidean parallel axiom together with all the other axioms of (plane) Euclidean geometry. Consequently, hyperbolic geometry satisfies the following parallel property: given a line ℓ and a point P not on that line, there is more than one line through P not meeting ℓ . Unlike the Euclidean plane and the sphere, the entire hyperbolic plane cannot be isometrically embedded in 3-dimensional Euclidean space. Therefore, any model of hyperbolic geometry in Euclidean 3-space must distort distance.

The *Poincaré circle model* of hyperbolic geometry has two properties that are useful for artistic purposes: it is conformal (i.e. the hyperbolic measure of an angle is equal to its Euclidean measure), and it lies within a bounded region of the Euclidean plane — allowing an entire hyperbolic pattern to be displayed. The “points” of this model are the interior points of a *bounding circle* in the Euclidean plane. The (hyperbolic) “lines” are interior circular arcs to the bounding circle, including diameters. The edges of the curved

triangles in Figure 1 and the backbones of the fish in Figure 2 represent hyperbolic lines.

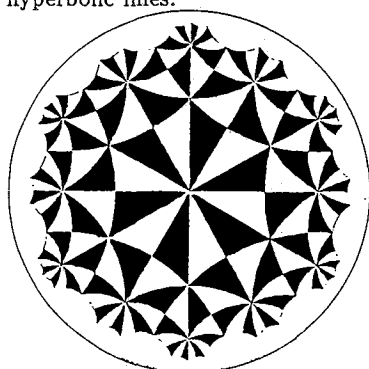


Figure 1. A pattern with symmetry group $[6, 4]^+$.

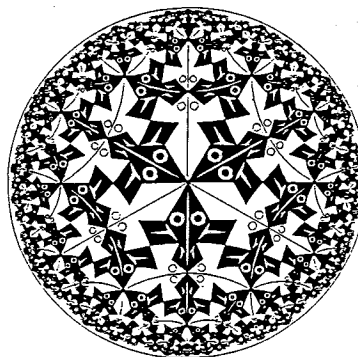


Figure 2. A computer generated rendition of M. C. Escher's Circle Limit I pattern.

Repeating Patterns, Tessellations, Symmetries

A *repeating pattern* of the hyperbolic plane is a pattern made up of hyperbolically congruent copies of a basic subpattern or *motif*. For instance, any adjacent black-white pair of triangles of Figure 1 forms a motif. Similarly, a black half-fish plus an adjacent white half-fish make up a motif for Figure 2.

An important kind of repeating pattern is the *regular tessellation*, $\{p, q\}$, of the hyperbolic plane by regular p -sided polygons, or p -gons, meeting q at a vertex. It is necessary that $(p - 2)(q - 2) > 4$ to obtain a hyperbolic tessellation. Figure 3 shows the tessellation $\{6, 4\}$ (solid lines) and its dual tessellation $\{4, 6\}$ (dotted lines).

A *symmetry operation* or simply a *symmetry* of a repeating pattern is an isometry (hyperbolic distance-preserving transformation) of the hyperbolic plane which transforms the pattern onto itself. For example, reflections across the backbones in Figure 2 and across any of the lines of Figure 3 are symmetries of those patterns (reflections across hyperbolic lines of the Poincaré circle model are inversions in the circular arcs representing those lines [or ordinary Euclidean reflections across diameters]). Other symmetries of Figure 2 include rotations by 180 degrees about the points where the trailing edges of fin-tips meet, and translations by four fish-lengths along backbone lines (in hyperbolic geometry, as in Euclidean geometry, a translation is the product of reflections across two lines having a common perpendicular, and the product of reflections across two intersecting lines produces a rotation about the intersection point by twice the angle of intersection).

The *symmetry group* of a pattern is the set of all symmetries of the pattern. The symmetry group of the tessellation $\{p, q\}$, denoted $[p, q]$, can be generated by reflections across the sides of a right triangle with acute angles of $180/p$, and $180/q$ degrees; i.e. all symmetries in the group $[p, q]$ may be obtained by successively applying a finite number of those three reflections. Thus, $[6, 4]$ is the symmetry group of the tessellation $\{6, 4\}$ formed by the

solid lines of Figure 3 — in fact $[6, 4]$ is the symmetry group of the entire pattern of Figure 3. The orientation-preserving subgroup of $[p, q]$ consisting of symmetries made up of an even number of reflections is denoted $[p, q]^+$. The symmetry groups of Figures 1 and 4 are $[6, 4]^+$ (it is just $[6, 4]$ if the color of the triangles is ignored), and $[5, 5]^+$ respectively. For more about the groups $[p, q]$, see Sections 4.3 and 4.4 of [Coxeter and Moser, 1980].

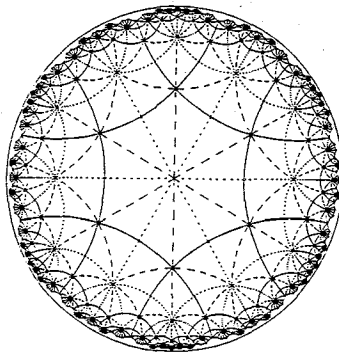


Figure 3. The tessellations $\{6, 4\}$ (solid lines) and $\{6, 4\}$ (dotted lines), and other lines (dashed) of reflective symmetry of the pattern.

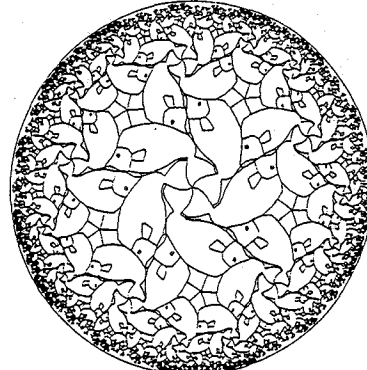


Figure 4. A pattern with symmetry group $[5, 5]^+$.

Color Symmetry

A pattern is said to have n -color symmetry if each of its motifs is drawn with one of n colors and each symmetry of the pattern maps all motifs of one color onto motifs of another (possibly the same) color; i.e. each symmetry permutes the n colors. The pattern of Figure 1 has 2-color symmetry (as does the Euclidean checkerboard pattern): reflection of the pattern across the side of any triangle interchanges black and white; rotation about a triangle vertex through twice its angle produces the identity permutation — black triangles go to black triangles and white triangles go to white triangles. For more on color symmetry, see [Senechal, 1983], and [Shubnikov and Koptsik, 1974].

Relation to Euclidean and Spherical Patterns

If the “ $>$ ” in the relation $(p - 2)(q - 2) > 4$ is replaced by “ $=$ ” or “ $<$ ”, one obtains tessellations of the Euclidean plane and the sphere respectively. In the Euclidean case, the corresponding symmetry groups of these tessellations are $[4, 4] = p4m$ and $[3, 6] = p6m$ which contain all 17 of the plane crystallographic groups as subgroups (see Section 4.6 and Table 4 of [Coxeter and Moser, 1980]). In the spherical case, the groups $[2, q]$, $[3, 3]$, $[3, 4]$, and $[3, 5]$, contain all the discrete spherical groups as subgroups. The notion of hyperbolic color symmetry also specializes to the usual notions of Euclidean and spherical color symmetry. If a pattern has symmetry group $[p, q]$ (disregarding color), we have found that there are 5, 2, and 16 possible kinds of 2-, 3-, and 4-color symmetry respectively for that pattern. Some of these

kinds of color symmetry require that certain divisibility conditions hold for p and q (e.g. p must be even). Consequently, when the divisibility conditions are not met, these kinds of color symmetry cannot appear in Euclidean or spherical patterns with symmetry groups of the form $[p, q]$. However, in the case of hyperbolic patterns with symmetry groups of the form $[p, q]$, there are infinitely many values of p and q satisfying *all* the divisibility conditions for 2-, 3-, or 4-color symmetry.

The Hyperbolic Pattern-Creation Process

The present version of the computer program allows for the design of repeating patterns with color symmetry whose symmetry group is a subgroup of $[p, q]$ and whose motif lies within a p -gon of the corresponding tessellation $\{p, q\}$. The pattern-creation process consists of two parts: (1) design of the motif, and (2) replication of the whole pattern from the motif. The design of the motif is done most easily with a computer graphics input device such as a data tablet or mouse — the motif is outlined by a sequence of points entered by the input device and connected by line segments.

To replicate a pattern from a motif, first note that it is easy to replicate that part of a pattern within a p -gon of $\{p, q\}$ if that p -gon already has a copy of the motif within it — the copy of the motif is simply rotated about the center or reflected across lines through the center of the p -gon. The algorithm for replicating the whole pattern depends on the fact that the p -gons of $\{p, q\}$ form “layers”: the first layer is a p -gon centered in the bounding circle, and each subsequent layer is defined inductively as the set of p -gons having a common vertex (only) or edge with a p -gon from the previous layer. Then it is merely a matter of moving a copy of the motif from one p -gon to another (either from one layer to the next or within a layer), using appropriate elements of the symmetry group. For more details on the pattern-creation process, see [Dunham, 1986a].

Bibliography

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