THE PARTHENON HEIGHT MEASUREMENTS: THE PARTHENON SCALE WITH ROOTS OF 2

A. M. BULCKENS

Name: Anne M. Bulckens, Ph.D. (Architecture), (b. Herentals, Belgium).
Address: Bulckens, Jalan Intan Ujung 75, Jakarta Selatan, 12430, Indonesia
E-mail: annepaul@cbn.net.id
Fields of interest: Classical, Renaissance and Baroque Art and Architecture
Publications:
Bulckens, A.M. (1998) Did Plato Ever Refer to a Section in Extreme and Mean Ratio in his Writings?, **Journal of Transfigural Mathematics**, *3, no. 2, 27-31 and no. 3, 23-31.*
Bulckens, A.M. and Shakunle, L. (1998) Logic Numbers and the music of Greek Architecture, **Journal of Transfigural Mathematics**, *4, no.1, 21-47, 53-57.*
Bulckens, A.M. (1999) **The Parthenon's Main Design Proportion and its Meaning.** *[Ph.D. Dissertation], Geelong: Deakin University, 269 pp.*
Bulckens, A.M. (2001) The Parthenon's Symmetry, in **Symmetry: Art and Science**, *(Fifth Interdisciplinary Symmetry Congress and Exhibition of the ISIS-Symmetry, Sydney, July 8-14, 2001), no. 1-2, 38-41.*

Abstract: The Parthenon (447 - 438 BC) epitomises the glory of Athens at the height of Classicism. Although enticingly forming the subject of numerous works of research, the understanding of several aspects of this temple to Athena still remains tentative. Many scholars adhere to the view that the Parthenon was designed in accordance with a simple 4:9 ratio. The dissertation, The Parthenon's Main Design Proportion and its Meaning, offers new insights. Through an analysis of the specific dimensions of the temple's building parts, this thesis establishes the measurement of a Parthenon Module and Parthenon foot (with 16 Dactyls). It then explains the main design proportion of the temple, expanding the idea that the Parthenon was built with a 4:9 ratio The measurements and design of the Parthenon's building elements exalt a 4:6 = 6:9 geometric proportion. This proportion involves the ratio 2/3, which is the perfect fifth in the Pythagorean tetrachord. It is proposed here that the design of the Parthenon, in regard to its height measurements, was based on a Dorian musical scale, which the architects altered in order to incorporate simple ratios representing the first roots of 2. In the next part of the article, not presented here, it will be shown how the Parthenon design exemplifies these ratios (that closely approximate roots of 2 values) two-

dimensionally, as well as three-dimensionally. Since it will then be revealed how the subtle design of the Parthenon houses two cubes, with the large cube having double the volume of the small cube, it then becomes clear that the Parthenon offers an answer to 'the Delian riddle of the cube', which at the time was a hot mathematical issue concerning $3\sqrt{2}$.

1. THE HEIGHT MEASUREMENTS OF THE PARTHENON

In the façade and flank, a total of twelve different height measurements are identified as having importance. These measurements are listed below, and they are shown in figures 1 and 2.
All measurements are based on the dimensions provided by Penrose [7], since it is well known that he was the leading archaeologist who measured the Parthenon in the most accurate and thorough manner. Penrose's measurements were then translated into a system of Parthenon Feet consisting of 16 Dactyls (D), whereby each Dactyl measures 21.44 mm and each Module measures 40 Dactyls [See Bulckens, dissertation, 1 and article, 2].

320 D = at the corner of the façade, acroterium + entire pediment (i.e. pediment face + raking cornice + rise in curvature of the horizontal cornice)
440 D = corner column minus the stylobate rise along the flank, minus the capital
480 D = column proper = central column height minus the stylobate rise along the flank
512 D = in the centre of the façade, upper step + column
560 D = in the center of the façade, 3 steps + column
623 1/3 D = at the corner of the façade, 3 steps + column + architrave
640 D = at the corner and also in the center of the façade, column + entablature
666 2/3 D = at the corner of the façade, 2 steps + column + architrave + frieze + its rise in curvature
720 D = in the center of the flank, 80 Dactyls of steps + column + entablature
800 D = in the center of the façade, column + entablature + pediment face
840 D = from the corner of the stylobate, column + entablature + pediment
960 D = from the corner of the stylobate, column + entablature + pediment + acroterium

Only two measurements are no integers: measurements 623 1/3 D and 666 2/3 D. The fractions of these measurements indicate that the Dactyls are divided further in thirds of Dactyls. As will be explained in this article, these two measurements were incorporated to bring out root of 2 values.

Clearly, all other Dactyl measurements are integers, yet *all* these Dactyl measurements are very accurate conversions of Penrose's dimensions into these Dactyl measurements, working well within a margin of 0.15 %. (It is noted that the 320 D measurement comprises the 200 Dactyl pediment (0.03 % off Penrose's dimension) plus the height of the acroterium that does not survive. Likewise, the 960 D measurement constitutes the 840 D measurement (0.03 % off) plus the same acroterium. An acroterium height of 120 Dactyl is a valid supposition; it is based on drawings by Orlandos.) Surmising, then, that these accurate integers indicate that the Parthenon designers did their utmost to obtain ideal measurements, it will be explored which design intentions underlie these ideal measurements.

It might well be that it is the Athenian conviction that they were especially blessed by the gods, hence superior of all others, which led the Athenians to demonstrate that this was indeed so. The astonishing level of perfection achieved by the architects and builders, as is evident in the Parthenon's design and construction, puts the 'self-image' of Periclean Athens on show – for the whole world to admire.

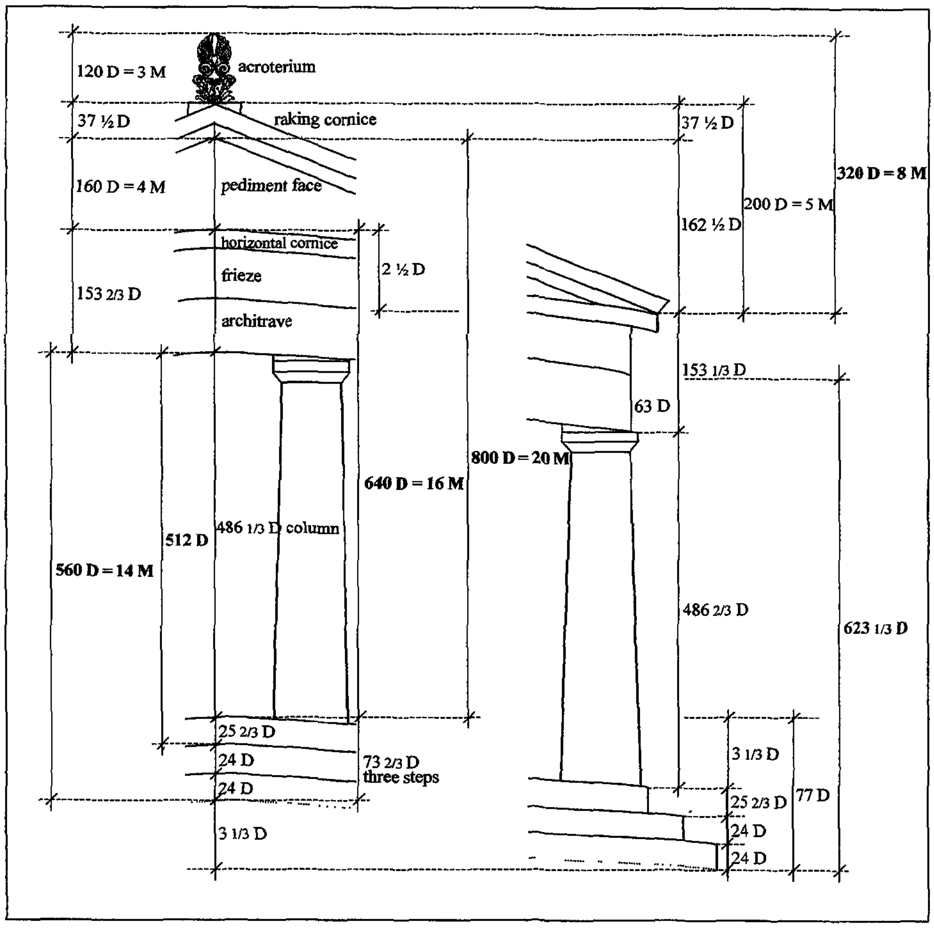

Figure 1: Parthenon diagram of the north-east corner and center of the façade, with curvature and steps exaggerated; drawing not to scale; dimensions indicated to the right of their dimension line.

One of the key measurements is the 480 Dactyl height of the 'column proper', which is the height of the central column of the façade minus the stylobate rise in curvature along the flank. (The stylobate is the temple platform upon which the columns stand. As with all 'horozontal' elements of the Parthenon, it is curved.) This measurement cannot be seen, but it will be explained how it plays a crucial role. Among the more tangible measurements at the center of the façade (shown in figure 1) are the 512 Dactyl measurement comprising the upper step plus the central column (25 2/3 D + 486 1/3 D), and the 560 Dactyl measurement consisting of the three steps and the central column (73 2/3 D + 486 1/3 D). Figure 1 shows also the 640 Dactyl measurement running from the stylobate center to the top of the horizontal cornice (central column + entablature, 486 1/3

D + 153 2/3 D), and the 800 Dactyl measurement running from the stylobate center to the top of the pediment face, excluding the perpendicular height of the raking cornice (central column + entablature + pediment face, 486 1/3 D + 153 2/3 D + 160 D).

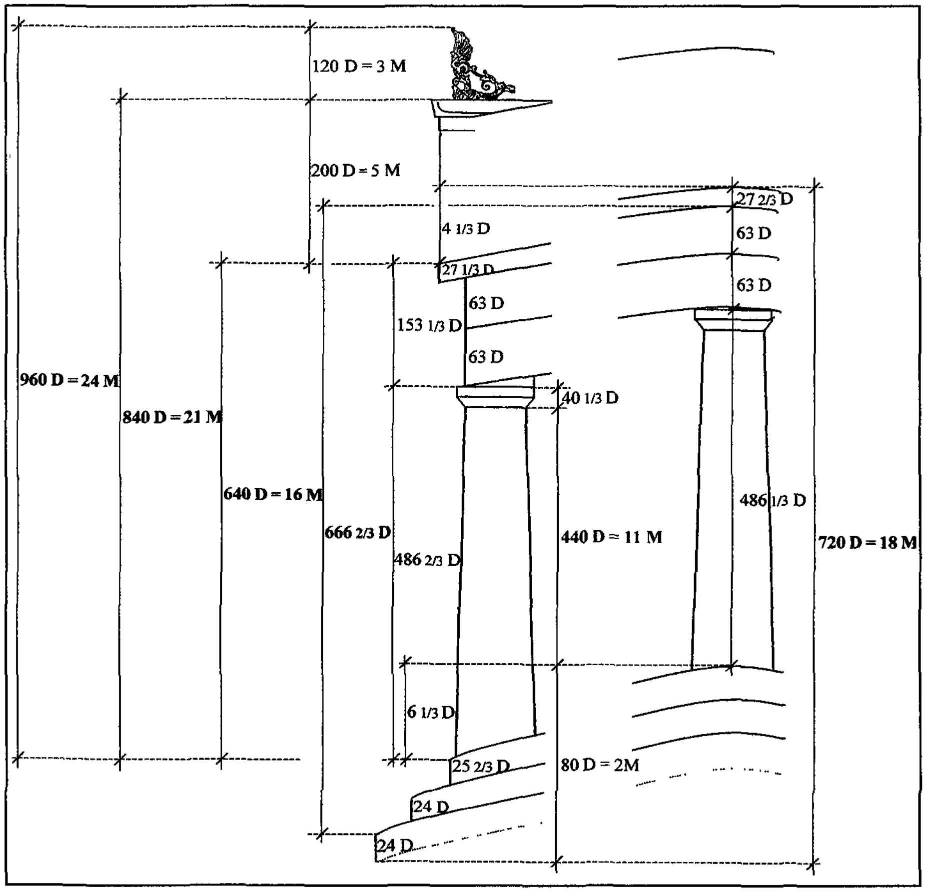

Figure 2: Parthenon diagram of the north-east corner and center of the north flank, with curvature and steps exaggerated; drawing not to scale; measurements of the 'Parthenon scale' in bold color.

In the center of the flank, shown in figure 2, is the 720 Dactyl measurement, which goes from the very bottom of the steps to the top of the horizontal cornice, including its rise in curvature along the flank (steps + stylobate curvature at the flank + central column + entablature, 73 2/3 D + 6 1/3 D + 486 1/3 D + 153 2/3 D). The 640 Dactyl measurement of the column and the entablature can also be taken at the corner (corner column + entablature, 486 2/3 D + 153 1/3 D). At this corner, the 623 1/3 Dactyl measurement (shown in figure 1), runs from the bottom of the steps to the top of the architrave (three steps + corner column + architrave, 73 2/3 D + 486 2/3 D + 63 D). And the 666 2/3 Dactyl measurement (shown in figure 2) starts at the top of the lowest step and reaches the top of the frieze, including the frieze's rise in curvature at the flank (second step + upper step + corner column + architrave + frieze + the frieze's rise in curvature at the flank, 24 D + 25 2/3 D + 486 2/3 D + 63 D + 63 D + 4 1/3 D). Lastly, the height from the corner of the stylobate to the top of the pediment is 840 Dactyls (corner column + entablature + pediment, 486 2/3 D + 153 1/3 D + 200 D), and to crown the temple, the acroterium height is added to this measurement, resulting in an overall temple height of 960 D.

2. TWELVE LARGE MEASUREMENTS DETERMINING THE HEIGHT OF ALL THE BUILDING ELEMENTS.

In the extended article, it will be explained in detail how these twelve measurements determine the height of *all the building elements of the façade and flank*: the three steps, the column and its capital, the entablature with its frieze, architrave and horizontal cornice, the pediment with its pediment face and raking cornice, and the acroterium.

It is proposed that these measurements determine the heights of the building elements in a peculiar way: when a large measurement is subtracted from a related larger measurement, the height of the building element in between these two measurements is obtained. And it is precisely the dimensions of the rise in curvature of the 'horizontal' elements, such as the stylobate and the entablature, which facilitate ideal dimensions and simple ratios between large measurements. The reason for this method will become clear in later paragraphs, but the method itself is easy to understand when giving a few examples. Consider measurement 800 and measurement 640, which are both taken at the center of the façade, starting from the top of the stylobate (the curved platform upon which the columns stand). Take the largest measurement of 800 Dactyls, and notice that it goes to the top of the pediment face. Subtract from this the 640 Dactyl measurement, which only reaches the top of the entablature, including the rise in curvature of its highest element, the horizontal cornice. What is left is a 160 Dactyl height. This is the height that results from the subtraction (800 – 640), and it determines the height of the pediment face because this is the building element between the two large measurements. Equally important, the equation 800 x 4/5 = 640, establishes the relationship, i.e. the ratio, between the two large measurements. Indeed, with this method of subtraction between large measurements, the height of *all* the building elements will eventually be determined, while simple ratios underlie the relationship between the large measurements.

Continue the same sequence and procedure with the 640 Dactyl height running from the center of the stylobate to the top of the entablature. Subtract from this the 486 1/3 Dactyl measurement comprising the central column. The result is the height of the entablature: it measures 153 2/3 Dactyls. Interestingly, as explained on previous occasions [1], it is the temple refinements, such as the rise in curvature of building elements, which facilitate a system of simple ratios between measurements. This is also the case with the height measurements, of which 640 cannot be readily multiplied by a simple ratio in order to obtain 486 1/3. Instead, it is necessary to incorporate the stylobate rise, in this case along the flank, to understand the measurement. The rise in curvature along the flank is 6 1/3 Dactyls. The column proper, i.e. the height of the central column minus the rise in curvature along the flank, measures 480 Dactyls. When the 486 1/3 Dactyl height of the column is thus split up in the 6 1/3 Dactyl height of the stylobate rise along the flank and the 480 Dactyl height of the column proper, the result is that: (640 x 3/4) = 480. Thus, when 640 is multiplied by a simple 3/4 ratio, the column proper is the result. And to obtain this column proper, a stylobate rise along the flank of 6 1/3 Dactyls was established and then subtracted from the column height.

In this fashion, the height of *all* the building elements will eventually be determined in the extended article. It will also be shown that there is a main sequence of measurements, which are related to each other by a continuous series of the ratios 1/2, 2/3, 3/4, 4/5, 5/6.

Consider the next sequence of measurements, which starts with the 480 that originates from the first sequence, and compare it to measurement 512. The 512 Dactyl measurement, taken in the center of the façade, is the height of the upper step plus the central column. Take the 512 Dactyl measurement and subtract from this the 480 Dactyl column proper, which is the height of the central column minus the 6 1/3 Dactyl stylobate rise in curvature along the flank. The result is 32 Dactyls. Hence, the height of the upper step is (32 – 6 1/3) Dactyls, which is 25 2/3 Dactyls. Again, a simple ratio forms the basis, since 512 x 15/16 = 480. And by subtracting the stylobate rise along the flank, I was able to determine the height of the upper step.

I propose that the temple was not only built with tremendous care, but that the architects already designed the temple adhering to this method, which allows working with a very small margin of error, always below 0.2 %. I propose that this is one of the reasons why the architects determined heights of building elements by taking simple ratios of larger measurements, while equating the 'left over', i.e. the subtraction result, with the height of the building element. This way, the architects could decide that, for example, 480 x 16/15 = 512, so that with this method employing 480, and a *simple* ratio, and 512, all works out correctly. On the other hand, if they would equate the height of the upper step, which in reality measures 25 3/4 Dactyls, with 25 2/3 Dactyls, the height of the step is off by a margin of error of about 0.3 %. This example shows the typical order of error of the building heights of the smaller elements, and what it makes clear is that: if it would solely be the height of each building element from which design ratios would be issued, the architects would not have enough leeway in order to form a coherent and logic design with simple ratios. When trying to find accurate simple ratios for the height of the upper step, the architrave, the frieze and the horizontal cornice, it becomes clear that there are no such accurate relationships within the design of the façade or flank.

Moreover, small measurements easily produce larger margins of error. On the other hand, *if the architects worked with the large measurements as the ratio generators, and if they employed the increments of the rises in curvature in various ways, they achieved the necessary leeway to make an accurate, coherent and complete design from simple ratios.*

In the extended article, the heights of all the building elements of the façade are established by working with twelve large measurements, simple ratios and the rises in curvature.

The only *temple* height not yet determined by these twelve large measurements is the height of the 2 steps of the cella. Beyond the façade, in the center of *the cella*, a 600 Dactyl measurement goes from the cella floor to the top of the entablature, excluding the entablature's rises in curvature. So, here is one more measurement that will be considered. Comparing the 640 Dactyl measurement taken at the corner, and the 600 Dactyl measurement taken in the center of the cella, the equation is 640 x 15/16 = 600, (This equation employs the same ratio as equation 480 x 16/15 = 512.) The difference is 1 Module, which comprises the height of the 2 cella steps and the rise of the upper floor in the 2 directions of the temple. (In fact, there is also a 600 Dactyl measurement, with 599 1/3 being 0.11 % off 600, at the corner of the façade going from the second step to the top of the architrave.)

3. THE PARTHENON SCALE WITH ROOTS OF 2

When the 600 Dactyl measurement is added to the height measurements, one gets a list of all the large measurements needed to obtain the heights of the main building elements *of the temple*. Measurement 512 can even be omitted, since it is possible, however cumbersome, to arrive at all building heights without this measurement. Thus measurement 512 is replaced by measurement 600, which also involves a ratio of 15/16 (the ratio of an enlarged half tone in music). (Concerning the ratios, the simplest route was taken to arrive at the heights of all building elements. Once many of the heights of building elements were determined, there were alternative routes on several occasions, which were not explained. These alternative routes provide the remaining ratios found between the heights in the 'Parthenon scale' shown in figure 3.)

When these twelve large height measurements are arranged in a linear scale as in figure 3, the temple heights reveal an elegant sequence of the ratios of the first ten integers, 10/9, 9/8, 8/7, 7/6, 6/5, 5/4, 4/3, 3/2, 2/1. Furthermore when 666 2/3 D is set as 100, it becomes clear that most of the heights are the same numbers as the tones of the basic Hindu-Greek musical scale [McClain, 5], later known as the 'reverse Dorian scale' and currently the common scale: do re mi fa sol la ti do [Kappraff, 3]. This Dorian scale is:

72	80	90	96	108	120	135	144
C	D	E	F	G	A	B	C

Basically, to arrive from the 'reverse Dorian musical scale' to the 'Parthenon scale', 80 is enlarged to 84 and 135 is reduced to 126. The result is that the intervals from C to D and B to C are enlarged whole tones, 7/6 and 8/7. When next 93 1/2 and 100 are inserted, the 'Parthenon scale' is obtained. Interestingly, according to Kappraff, a musician-mathematician, 100 falls on Fsharp [Kappraff, 3].

72	84	90	93 ½	96	(100)	108	120	126	144
C	D	E		F	Fsharp	G	A	B	C

In figure 3, when 666 2/3 Dactyls are normalized to hundred, all Dactyl measurements are divided by 6 2/3, so that all the numbers that are the same as the ones from the Dorian scale result in integers, while their ratios become readily readable. As in the Pythagorean tetrachord, 72–108 and 96–144 form perfect fifths, while 72–96 and 108–144 form perfect fourths, and 640–720 forms a whole tone. Surprisingly, 84–126 is a 3/2 ratio also. When this ratio is split up to reach the 100 in between 84 and 126, the fourth root of 2 (which here is 100/84) and the cube root of two are obtained. In this case, the cube root of 2 is the old Babylonian 126/100. (According to McClain [6], the old Babylonian cube root of 2 was regarded as 125/100 + 1/100.)
In all, the only numbers that differ from the Dorian scale, are 126 and 84, while 93 ½ and 100 are inserted. These are the tones, which create ratios that closely approximate the roots of 2. So, by altering the Dorian musical scale in a unique manner, the 'Parthenon scale' is created to provide the first roots of 2.
The *column height* is crucial in the 'Parthenon scale'. The 480 D height of the column proper forms the beginning of a musical octave. Its double is the overall temple height, and this is exactly where an octave ends. Also, the column height assists in providing the measurements 623 1/3 and 440. The ratio between these two measurements is 17/12, which is a Pythagorean approximation of $\sqrt{2}$. [Lasserre, 4, and Kappraff, 3].

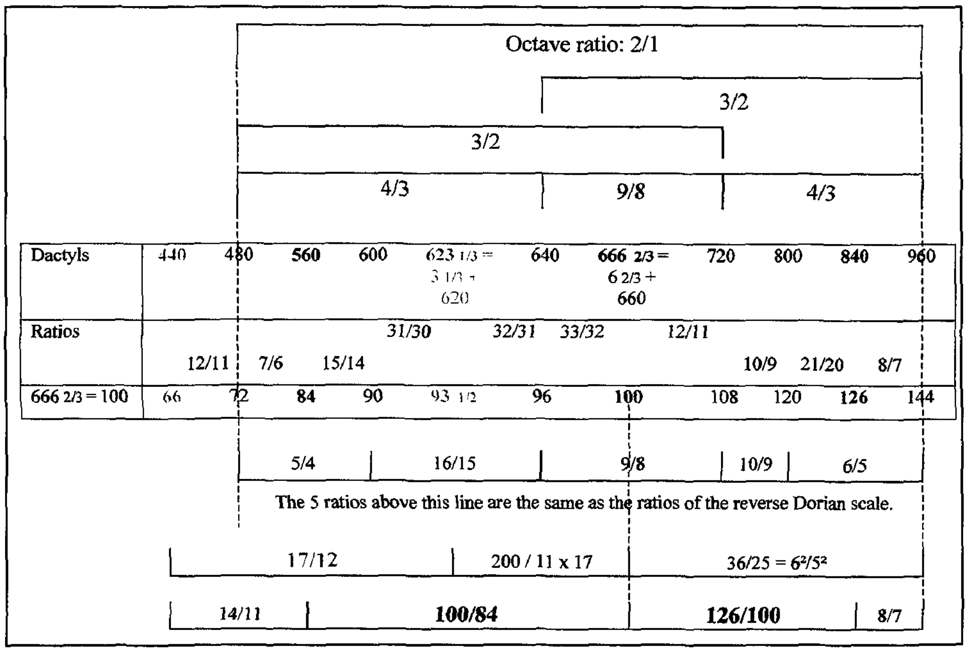

Figure 3: The'Parthenon scale'; the 480 – 600 – 640 – 720 – 800 – 960 is the reverse Dorian scale, the ratio between 840 and 666 2/3 represents $^3\sqrt{2}$; the ratio between 666 2/3 and 560 represents $^4\sqrt{2}$; and the ratio between 623 1/3 and 440 represents $\sqrt{2}$.

In conclusion, by creating the 'Parthenon scale', simple ratios are revealed between numbers, of which 72 – 96 – 108 – 120 – 144 are exactly the same as the tones in the reverse Dorian scale. There are more similarities between the 'Parthenon scale' and musical scales; In Pythagorean musical scales, the tetrachord is the framework and the ratios between other tones could vary, depending on the 'mode' that was adhered to. With these other tones in between, there came Pythagorean commas, Pythagorean leimnas, etc, which in essence are 'left-overs' that had to cover the distance to reach the octave. This is also the case in the height measurements of the 'Parthenon scale'. The height of some building elements are literally 'what is left over' after subtraction between large measurements. The difference between historic musical scales and the 'Parthenon scale' is that the 'Parthenon scale' incorporates ratios, which turn out to be excellent approximations for the roots of 2, whereas in the Pythagorean musical scales, $\sqrt{2}$ is the geometric mean of the octave, but it could never be reached by a coherent system of simple ratios. (The 126/100 ratio is less than 0.1 % off the real value of $^3\sqrt{2}$, while the 17/12 ratio is 0.17 % off the real value of $\sqrt{2}$.)

It needs to be stressed that the 'Parthenon scale' was not made out of the blues! On the contrary, every single number of the scale accurately refers to a height measurement of the Parthenon. And as it is shown at length that the height of each building element could be determined by these twelve measurements and their interrelated ratios, I suggest that the architects had exactly the same measurements and ratios in mind in order to design the heights of all the building elements of the façade and flank.
This article shows how the Parthenon design is rooted in Pythagorean music, whereby Kappraff and McClain contributed greatly to the understanding of ancient musical scales and theory. Concerning the Parthenon, the design issue was to marry a set of measurements exalting roots of 2 with a set of measurements, which were known to form part of the Dorian musical scale. When the 666 2/3 Dactyl measurement was normalized as 100, it provided the means to merge the two sets together in such a handsome way, that even the complete sequence of the ratios 10/9, 9/8, 8/7,..., 2/1 was the result. In the next article, then, it will be explained *why* and *how* the architects incorporated the roots of 2, especially $\sqrt[3]{2}$, in the Parthenon's plan and elevations.

References

Bulckens, A.M. (1999) **The Parthenon's Main Design Proportion and its Meaning.** [Ph.D. Dissertation], Geelong: Deakin University, 269 pp.

Bulckens, A.M. (2001) The Parthenon's Symmetry, in **Symmetry: Art and Science,** *(Fifth Interdisciplinary Symmetry Congress and Exhibition of the ISIS-Symmetry, Sydney, July 8-14, 2001), no. 1-2, 38-41.*

Kappraff, J. (2002) Anne Bulckens' System of Proportions of the Parthenon and its Meaning. A presentation at the Carleton University School of Architecture, Ottawa.

Lasserre, F. (1964) **The Birth of Mathematics in the Age of Plato.** *London. Hutchinson.*

According to Lasserre, Plato used the equation $x^2 - 2y^2 = \pm 1$, to find ratios between integers x and y that closely approximate $\sqrt{2}$. He further asserts that Plato, when using this equation, was Pythagorising. Since $17 \times 17 - 2 \times 12 \times 12 = 1$, it follows that 17/12 approximates $\sqrt{2}$, and thus according to Lasserre, the Pythagoreans knew this.

According to Kappraff, the ratio 17/12 was also used in Roman architecture (Ostia), in order to approximate $\sqrt{2}$.

McClain, E. (1976) The Myth of Invariance: The Origin of the Gods: Mathematics and Music from the Rg Veda to Plato. *York Beach: Nicolas-Hays.*

McClain, E.(1997) The 'Star of David' as Jewish Harmonical Metaphor, **International Journal of Musicology**, no. 6, 43-47.

Penrose, F.C. (1888) **An investigation of the Principles of Athenian Architecture.** *2nd ed., Washington, D.C: McGrath.*