GROWTH, CURVATURE AND COMPUTATION

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Publications:

Goodman-Strauss, C. (1998) Matching rules and substitution tilings, *Annals of Mathematics*. 147, 181-223.

Goodman-Strauss, C. (2000) Open questions in tilings, preprint.
Goodman-Strauss, C. (2002) Regular Production Systems and Triangle Tilings, preprint.

***Abstract**: We discuss some new mathematical techniques for describing growth and form across a range of mathematical, biological and artistic applications. The essential idea is in the air these days: to describe emergent complex behaviour through simple local interactions. These interactions are shaped by local combinatorial restrictions, and in turn give rise to global geometric properties and tremendously rich behaviour is possible. New tools are proving to be quite powerful, producing a body of interesting mathematical, but too they seem to shed light on real phenomena, such as the way that tissue grows to produce complex geometric forms such as the human ear. We will discuss the mathematical idea philosophically, with an emphasis on presenting a wide range of graphical applications.*

In this talk, I'd like to present a model of growth and form that arises from local interactions among little combinatorial agents. These agents can be thought of as puzzle pieces which may fit together in certain ways; two pieces can be neighbors only if they are compatible. To visualize this, imagine a sea filled with copies of a few kinds of the organisms in Fig 1: these organisms have hands of certain shapes and sizes that can fit together in only certain ways with each other. Even though it is very simple to tell what is happening *locally*, the global behaviour of the system may be quite subtle. The study of such agents immediately touches on fundamental issues in the theory of computation, and rich, beautiful behavior arises.

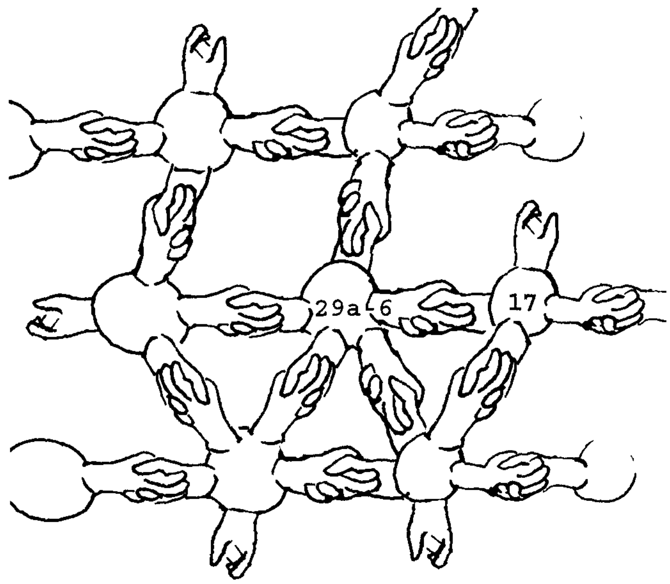

Figure 1: A cartoon model of local combinatorial structure.

We will focus on how **curvature** can arise through these interactions; in particular, we will see that there will be no method, *a priori*, of determining just what the global curvature of a given system will be. Conversely, all kinds of behaviors will be possible.

Curvature

Before getting underway, I'd like to take a moment to discuss the curvature of surfaces. These ideas are well known and quite old, but may be unfamiliar to the general audience. A ***flat*** surface is a good place to start (Fig 2): a flat surface has the same local metric properties as the Euclidean plane. In particular, as a circle grows on a flat surface, the circumference grows *linearly* with as the radius of the circle increases. The cylinder, for example, is in fact a flat surface. (An easy way to see this is simply to roll up a sheet of paper). This rasies an important point: "flatness" and curvature are *intrinsic* properties of the surface. It doesn't matter one bit that a cylinder is "curvy": it's flatness depends on the way we measure things when we are confined to the surface.

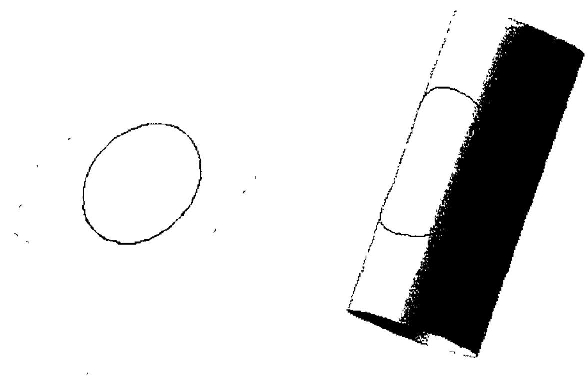

Figure 2: Two flat surfaces.

This isn't so strange-- the earlier image of the plane was in perspective, and distance was certainly not accurately rendered in that image either. In Figure 3 we see two images of a checkerboard. The images appear quite different but they are *intrinsically* the same, measuring distances so that all the squares are to be of uniform size. An inhabitant would not be able to tell which of the two spaces he lived in.

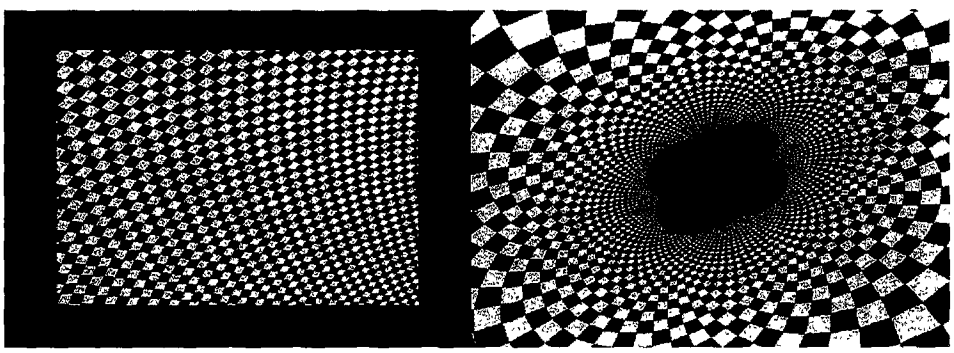

Figure 3: Geometry is intrinsic; two more flat surfaces.

We can contrast this with **positively curved** surfaces, such as the sphere, and **negatively curved** surfaces such as a crinkly piece of lettuce.

A sphere really is curved; there is no way to make a sphere out of flat pieces. On a sphere, the circumference of a circle grows much more slowly than the radius, and the amount of curvature can be measured by this "deficit". As before, the particular rendering is unimportant; the intrinsic notions of distance on the surface define curvature. We are quite used to seeing this: the Earth is not flat despite the image at right in Figure 4.

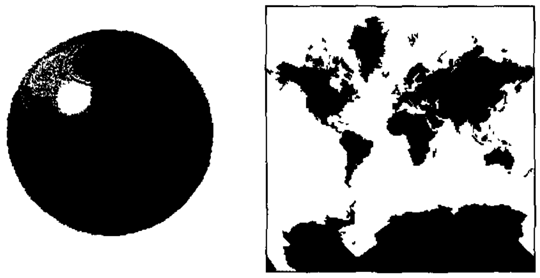

Figure 4: Positive Curvature.

On a negatively curved surface, the circumference of a circle grows exponentially as the radius increases. A physical object with negative curvature will typically be quite crinkly, or have numerous "tubes". In Figure 5, at middle at right, we see two forms, drawn by Ernst Haeckel, with high negative curvature.

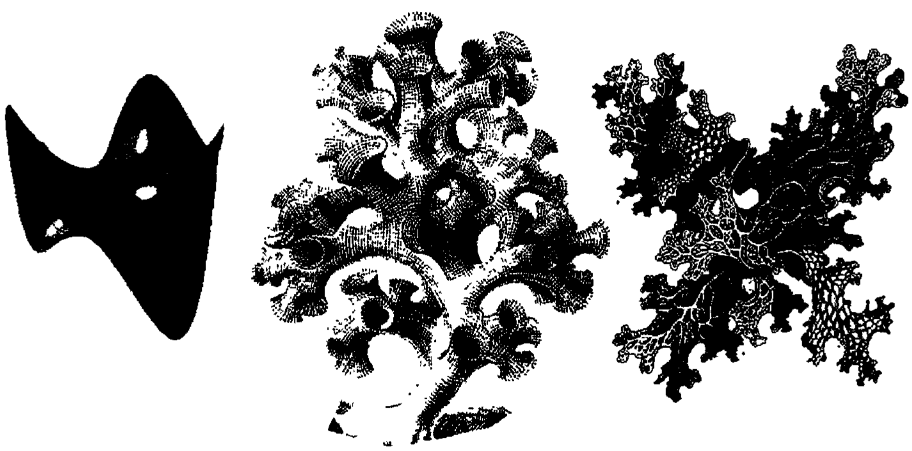

Figure 5: Negative Curvature.

Combinatorics and Curvature

Consider fitting together puzzle pieces to make a surface. For the moment, we will suppose we that the surface is "growing" outwards along some boundary. (Fig 6, again due to Haeckel)

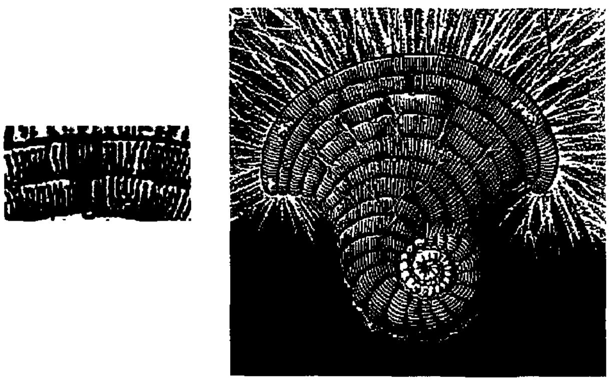

Figure 6: Growth of a surface along a boundary.

Locally the curvature of the resulting surface depends on how fast this boundary is expanding. If the boundary is expanding rapidly with each step, then the curvature will be negative. If the boundary is contracting, or growing very slowly, the curvature of the surface will be positive. This is precisely the effect we see here: as the shell accretes, the boundary is growing extremely rapidly and we have high negative curvature. This is precisely the effect we see in Figure 7: as the shell accretes, the boundary is growing extremely rapidly and we have high negative curvature.

Figure 7: Highly negatively curved surface.

One may enjoy performing a small experiment to see how local conditions give rise to curvature. Consider gluing many equilateral triangles together to make a tiling or polyhedron. If exactly six triangles meet at each vertex, the resulting surface will be flat. If fewer than six meet at each vertex, the result will be a surface with positive curvature (for example, if five meet at each vertex, we will obtain an icosahedron– a spherical surface). And if more than six meet at each vertex, we will obtain a surface with negative curvature. In fact, this is all true regardless of the specific geometry of the triangles used– what is important is how they meet, i.e. the nature of the local combinatorics.

Now suppose the local curvatures are uniformly mixed: in some spots the curvature is positive, and elsewhere negative. What will the overall curvature be?

If we can calculate the overall proportion of each local behavior we can easily calculate the overall curvature of the surface. For example, consider triangles that can meet in two kinds of ways: ***113*** and ***331*** (that is, in 5's and in 7's) There is in fact only one possible solution: the resulting surface must be flat: the two kinds of local curvature must balance out perfectly.

Combinatoria

I'd like to turn now to the kinds of local combinatorial objects I'm considering. Such gadgets can be thought of as little puzzle pieces that can fit with one another. One example of such gadgets are tiles in the Euclidean plane. Here, of course, we know *a priori* what the local curvature will be.

But consider the following question:

Given a set of tiles, can you tell whether you can cover the entire Euclidean plane with copies of these tiles? For example, can you cover the plane with copies of the tile at left in Figure 8? How about with copies of that at right? (These examples are due to Jarkko Kari).

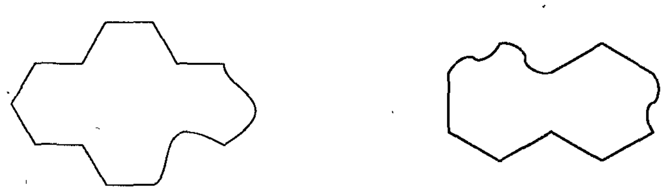

Figure 8: Which of these two tiles can admit a tiling?

This question is known as the "Domino Problem". The real question is:

Is there a way to decide the answer in general? That is,

Is the Domino Problem "decidable"?

The Domino Problem is exceedingly subtle, even when we have only one kind of tile. (As a hint of this, even the two tiles in Figure 8 are tricky to analyze. The tile at left in Figure 8 can tile a large region but no more (Figure 9). The tile at right in Figure 8 can tile, but the simplest way this can be achieved is fairly complicated (Figure 10).

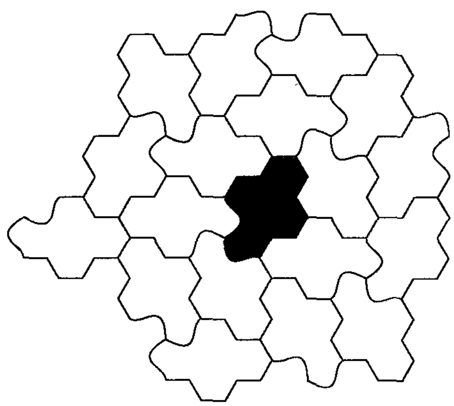

Figure 9.

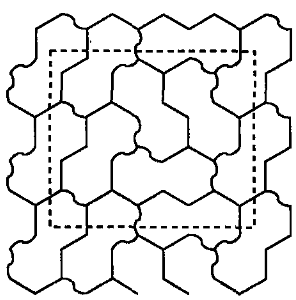

Figure 10.

In 1966, R. Berger showed the Domino Problem is *undecidable* in the Euclidean plane. That is:

There is no *general* method to decide, for a given set of tiles, whether they can form a tiling

Any computation can be modeled by some set of tiles.

A very nice corollary to this is that there exist *aperiodic* sets of tiles– tiles that can tile but can never tile periodically. The Penrose tiles are the most famous example (Figure 11).

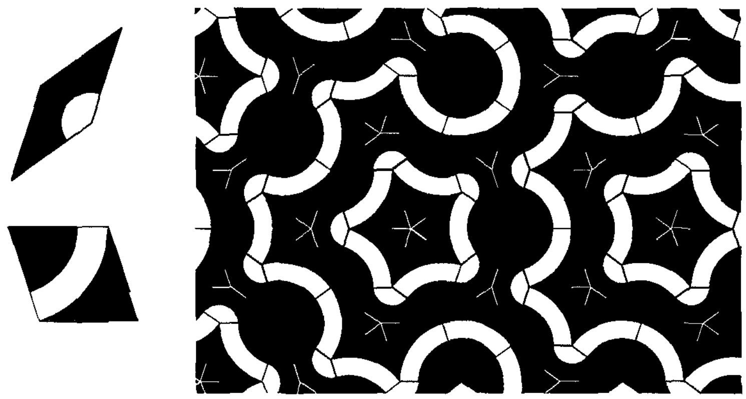

Figure 11: The Penrose rhombs.

Now for our game here we are laying down layer after layer of these combinatorial objects; can the overall rate of growth be determined?

Symbolic Subsitution Systems

A first model is symbolic substitution systems. One begins with an alphabet, say 0, 1, and a set of replacement rules, say 0→1, 1→10

This defines a map on the set of all words in this alphabet. For example:

0110 →1 10 10 1 and in turn 110101 → 1010110110

We can define *superwords*; these arise from applying this map repeatedly to our alphabet. So for example, here our superwords are

0→1→10→101→10110→10110101→1011010110110 etc

Now note:

a) Such symbolic substitutions can model the kind of growth we've been discussing. The "letters" describe our combinatorial objects. The "words" describe our boundaries.

b) Classical theorems describe precisely the rate at which word-lengths grow under the substitution, and the overall distribution of letters in the superwords.

In our example 0→1, 1→10 words tend to grow by the golden ratio and in the large, the ratio of the number of 1's to the number of 0's will also tend to the golden ratio.

In particular, word length grows exponentially.

The tiling of the hyperbolic plane shown in Figure 11 is precisely described by the system 0→1, 1→10. And indeed all symbolic substitution systems can be modeled in similar fashion.

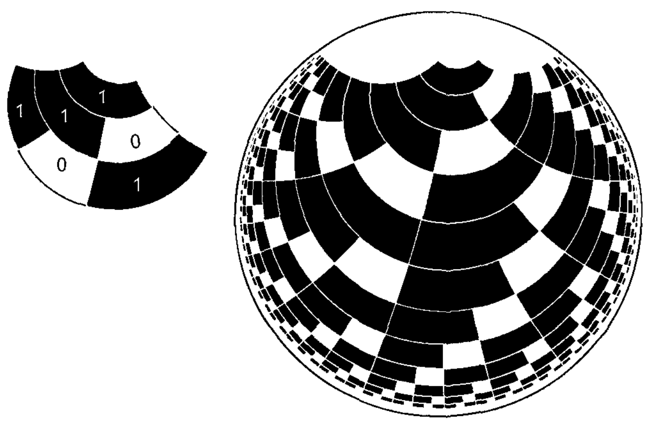

Figure 12: A tiling of (the Klein model of) the hyperbolic plane based on the system 0→1, 1→10.

Regular Production Systems

However, the general situation is much more subtle. In brief, one considers an alphabet as before. However we restrict ourselves to a "regular language" of allowed words. For example, let our alphabet be 0, 1, 2 and take the language of words where: 0 can only be followed by 1, 1 can only be followed by 2, and 2 can only be followed by 0 or 1. So for example, the words 1201 and 012120 are in the language, but 1120 is not.

For each letter we take one or more replacement rules. In this example we take

$0 \to 12$ $\quad$ $1 \to 12$ $\quad$ $2 \to 20$ $\quad$ $1 \to 21$ $\quad$ $2 \to 01$

Here the replacement is not deterministic. For two words W, V in our language, we write $W \to V$ if there is *some* choice of replacements on the letters that takes W to V. *But Note:* a given word may be mapped to one, no or many words. For example the word 012120 can only be mapped to 12 12 01 21 20 12; the word 0120 can't be mapped to any word at all. And the word 1212 can be mapped to either 12012120 or 21201201.

The point of these systems is that they can model any arrangement, with any curvature, of any set of combinatorial objects. The letters describe the pieces, the language how these pieces fit together in layers, and the rules how each layer may fit with the next. A cartoon of this is shown in Figure 13.

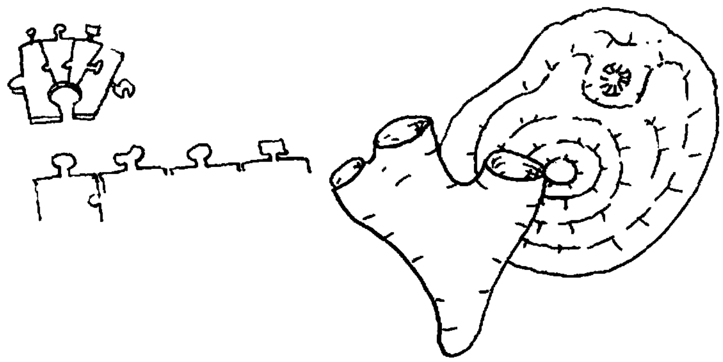

Figure 13: A cartoon illustration of how regular production systems can be used to model growth by the accretion of combinatorial elements along a boundary.

However, quite unlike the classical substitution systems, it is likely that it is undecidable, for example, whether one may repeatedly substitute ad infinitum; it is quite likely that it is undecidable whether a given rule will be needed; it is likely that it is undecidable how frequently a given rule will be applied.

In particular, it is certainly undecidable, given a particular regular substitution system, what the curvature of the corresponding surface will be.

Or to put it another way, any desired behaviour can be attained.

Post Tag systems and growth

I'd like to demonstrate this with an adaptation of an universal computer due to Emil Post, called "Post Tag Productions"

One has an alphabet, a set of rules and a starting letter:

a→abc	
b→a	abbca
c→ba	

At each step, one has a word. You cross off the first two letters, and depending on the original first letter, add a word to the back. So for example:

abbca→ bca abc (crossing off ab and adding abc)

bcaabc→ aabca→ bcaabc etc. This particular system repeats quite quickly. However, a little experimentation shows it is difficult to predict whether the words in a given system will eventually repeat, grow forever, or shrink away to nothing. I encourage the reader to try a few more examples– simply select different rules, a different alphabet, or a different starting word. In fact the long term behaviour is in general undecidable and indeed *any* computation can be modeled as a Post-tag system.

In a manner that we cannot explain here, this implies that the curvature induced by a regular production system (that is, how fast the words in aregular production system grow) is undecidable; correspondingly, anything can be achieved. This is precisely, in fact, why nature has access to such a staggering variety of forms through simple growth along a boundary, using elementary rules.

A complete bibliography and proper treatment of this material can be found in the papers "Open Problems in Tiling" and "Regular Production Systems and Triangle Tilings" both available at http://comp.uark.edu/~cgstraus/papers.

References

Berger, R. (1966) The undecidability of the domino problem, *Memoirs Am. Math. Soc.* 66 (1966).

Gardner, M. (1977) Extraordinary nonperiodic tiling that enriches the theory of tilings, *Scientific American* 236, 110-121.

Goodman-Strauss, C. (1998) Matching rules and substitution tilings, *Annals of Mathematics*. 147, 181-223.

Goodman-Strauss, C. (2000) Open questions in tilings, preprint.

Goodman-Strauss, C. (2002) Regular Production Systems and Triangle Tilings, preprint.

Grünbaum, B and Shepherd, G.C. (1987) *Tilings and patterns*, W.H. Freeman and Co.

Minsky, M.L. (1967) *Computation and infinite machines*, Prentice-Hall, Englewood Cliffs, N.J.

Queffelec, M (1987) *Substitution dynamical systems—spectral analysis, Lecture Notes in Mathematics* 1294, Springer-Verlag, New York.

Robinson, R.M, (1971) *Undecidability and nonperiodicity of tilings in the plane*, Inv. Math. 12, 177-209.

Thurston, W. (1989(*Groups, tilings and finite state automata*: Summer 1989 AMS Colloquim Lectures, unpublished notes.