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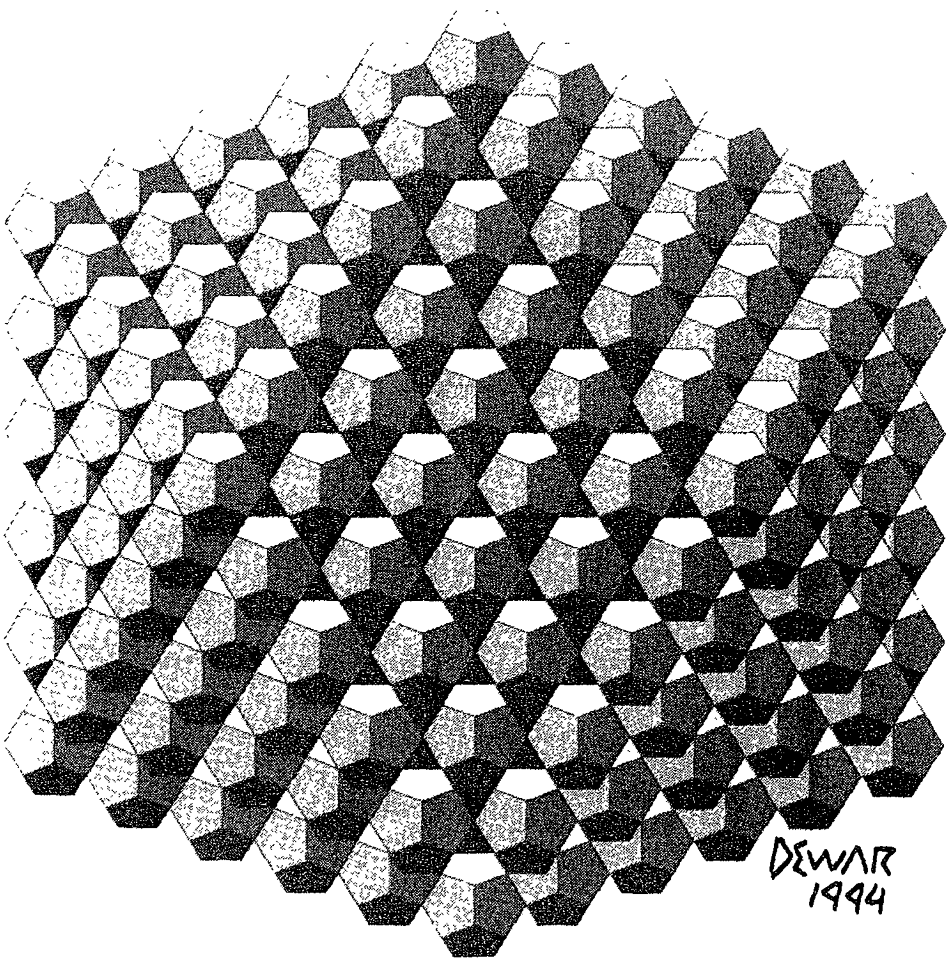

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ON THE VARIOUS PHASES OF CIRCLES・SPHERES IN WASAN

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The main objects of this paper are treated the problems about Descartes Circle Theorem and the related Wasan problems, and more the three dimensional extentions.

1. Short History of the Descartes Cercle Theorem.

Decartes Circle Theorem,[2] that is a theorem on curvatures of touched circles,[7] that is: When three circles(curvatures are e_1, e_2, and e_3) circumscribe each other and the fourth one (e_4) contacts the other three, respectively, there is the following relation (Fig.1, Fig.2).

$$(e_1 + e_2 + e_3 + e_4)^2 = 2(e_1^2 + e_2^2 + e_3^2 + e_4^2)$$

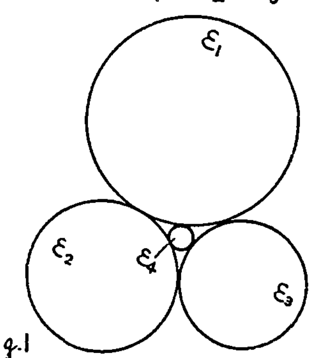


Fig.1

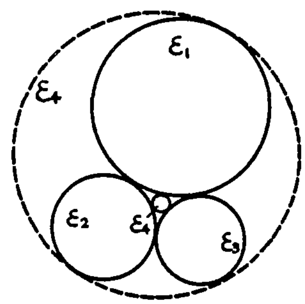


Fig.2

Remark. The Descartes Circle Theorem in Rene' Descrtes' complete works[2] do not shaped itself into a theorem, because it was his letter to Princes Elizabeth.[8]

Many scholars are extension this Theorem. That is, Steiner solved it on the curvature of the fourth circle:

$$e_4 = e_1 + e_2 + e_3 + 2(e_1 e_2 + e_2 e_3 + e_3 e_1)^{\frac{1}{2}} \tag{1}$$

Melzak[6] and Stanton[12] started the above formura, and they extended it to n-th circles. H.S.M. Coxeter[1] and Morley,[11] moreover Walker are extended the n-th circles. Substituting α, β, γ_1 and γ_1 for e_1, e_2, e_3 and e_4, respectively in Steiner's formula (1), he defined the n-th circle (curvature γ_n) inductively and he got the following:[14]

$$\gamma_n = \gamma_0 + 2n\delta +_1 n^2(\alpha + \beta)$$

where $\delta = (e_1 e_2 + e_2 e_3 + e_3 e_1)^{\frac{1}{2}}$ [1], that is $\delta = (\alpha\beta + \beta\gamma_0 + \gamma_0\alpha)$.

2. The study of it in Japan.

Nushizumi Yamaji(山路主住 1724 - 1772) wrote Zeishiki Endan 贅式演段 at 1751. In this, he says about the Descartes Cirle Theorem as follows:

When three circles having radii r_1, r_2 and r_3 circumscribe each other and fourth one (radius x) touches the other three, then there is following relation: [7]

$$(r_1 r_2 r_3)^2 - 2 r_1 r_2 r_3 (r_1 r_2 + r_2 r_3 + r_3 r_1)\, x + (r_1^2 r_2^2 + r_2^2 r_3^2 + r_3^2 r_1^2 - 2 r_1^2 r_2 r_3 - 2 r_1 r_2^2 r_3 - 2 r_1 r_2 r_3^2)\, x^2 = 0.$$

That is, we have rewritten as follows:

$$x^{-1} = e_4 = s_1' \pm 2\sqrt{s_2'},$$

where s_1', s_2' and s_3' are expressed the fundamental symmetric expression of e_1, e_2, e_3.

Theorem[7] Let a point O be the center of two concentric circle having curvatures ρ and 3ρ. Let circles O', O'' (curvature 3ρ) be in contact externally with each other and also with the two circles above. We use the same symbols O', O'' for the centers of these cirles.

Ler P_1 (σ_1) be the center (curvature) of a circle which is in external contact with the circle O', O'' and the inner circle O. Morover, let Q_1 (τ_1) be the center (curvature) of a circle which is in external contact with the circls O', O'' and is in internal contact with the outer circle O.

Furthermore, let σ_2 (τ_2) and P_2 (Q_2) be the curvature and the center of a circle which is in external contact with the circles O', P_1 (Q_1) and the inner (the outr) circle O, ... Repeating this proceses, are obtain the series σ_2, σ_3, ... , σ_n (τ_2, τ_3, ..., τ_n) respectively.

Then the following invariable relation is held (Fig.3):

$$3\,\tau_n - \sigma_n = 6\rho \;(n = 1, 2, \ldots, n).$$

Fig.3

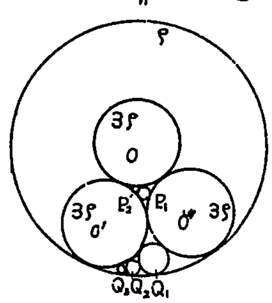
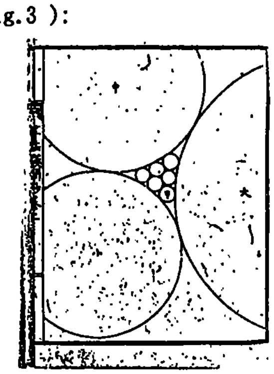


Fig.4



Recent:[5] When three circles A, B, C (radius a, b, c) circumscribe each other and more another three circles D, E, F circumscribe each other in that three circles A, B, C's gap. Then the forth circle's radius d is as follows (Fig. 4):

$$d = abc \,/\, bc + 2a\,(b + c) + 2 \left\{ 2abc\,(a + b + c) \right\}^{\frac{1}{2}}.$$

Remark 1. Let e_1, e_2, e_3, e_4 be the curvatures a^{-1}, b^{-1}, c^{-1}, d^{-1}, then we have by Wasan experts as following:

$$e_4 = e_1 + 2(e_2 + e_3) + 2\delta.$$

This is so called 大円傍斜術.

Remark 2. This problem is on the mathematical tablets in Soshidō, Bushū Horinouchi at Bunkyū 3 (1863)(Fig. 5).

3. The study of it in three demensions.

The Descartes Circle Theorem easy expand to three demension. But its omitt.

Theorem[10] A spherical segment has diameter a and hight b. Four equal spheres of diameter r are tangential to each other. Two of them are tangent to the spherical segment and the plane base, one is tangent with the spherical segment only and the last is tangential to the plane only (Fig. 6):

Theorem[7] The large sphere G having curvature ρ is packed with equal sphres having curvature 3ρ. A_1, A_2, A_3 and A_4 are four spheres of the above equal ones and they touch each other.

Let P_1 be a sphere circumscribed with the above four equal ones, and P_2 be one circumscribed with the spheres A_1, A_2, A_4 and $P_1, \dots$. Finally P_7 is defined as the same process, then P_7 agrees to P_1.

And more, let Q_1 touch the spheres A_1, A_2, A_4 and large one Q_1, Q_2 do the spheres A_1, A_2, Q_1 and large one G, ... Finally Q_7 is defined as the same process, then Q_7 agrees to Q_1.

Furthermore, let σ_n (τ_n) be the curvature of sphere P_n(Q_n), then

$$3\tau_n - \sigma_n = 6\rho \;(n = 1, 2, \dots, 6)$$

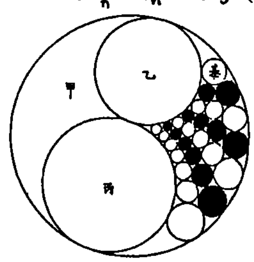


Fig. 5

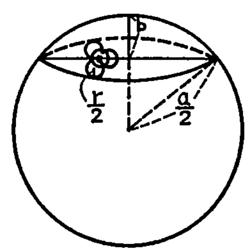


Fig. 6

4. The study of it in Japan (2)

Tansaku Sampō is wroted 1840 by Shokō Kenmochi (1790 - 1871)[8]. The seven problems like this are treated by Wasan experts. One of them as follows.

Problem[8] Three spheres A, B and C are circumscribed each other and they put between the flat boards. The fourth sphere D being touched the boards and the spheres A and B. Similarily the fifth sphere E and the sieventh one F in made. Now, the radiuses of the spheres A, B and C are 6, 3, 4 centimeters, then let seek the radiuses of the spheres D, E and F (Fig.7).

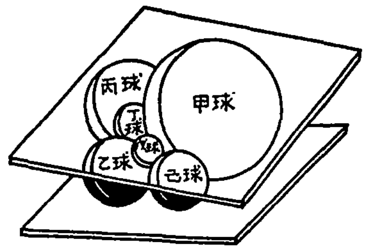


Fig.7

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