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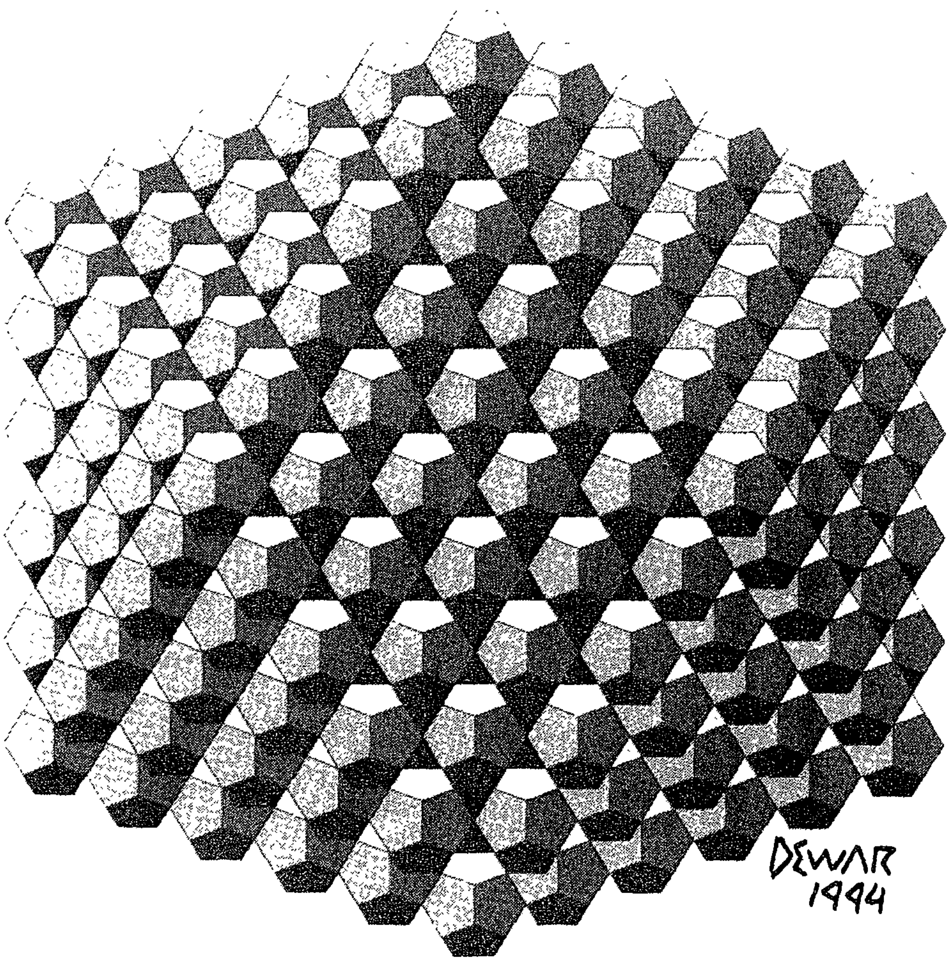


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**PACKINGS, COVERINGS AND BETWEEN**

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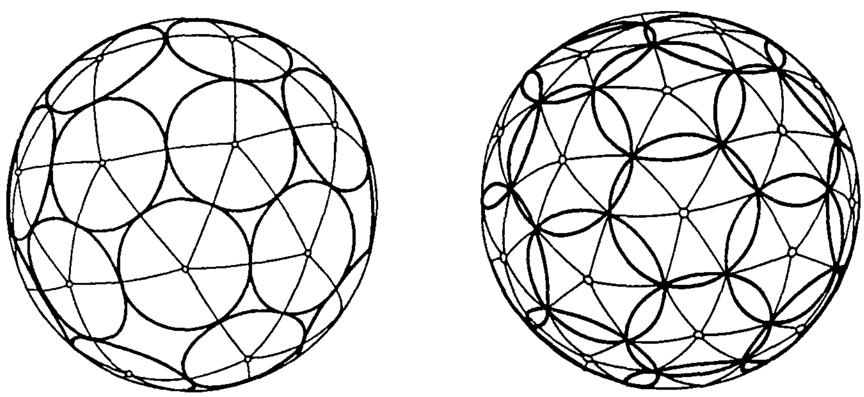
In the paper arrangements of equal circles on the spherical surface are discussed (Fejes Tóth, 1964). The Dutch botanist Tammes raised the following mathematical problem: How must $n$ equal circles (spherical caps) be packed on a sphere without overlapping so that the angular radius of the circles will be as large as possible? This *packing* problem has a dual counterpart in *covering*: How must a sphere be covered by $n$ equal circles (spherical caps) without interstices so that the angular radius of the circles will be as small as possible? Let $r_n$ and $R_n$ denote the maximum angular radius in the packing problem and the minimum angular radius in the covering problem, respectively. If we have circles with angular radius $r$ such that $r_n \leq r \leq R_n$ then a problem *intermediate* between these two can be posed: How must $n$ equal circles (spherical caps) of given angular radius $r$ be arranged on the surface of a sphere so that the area covered by the circles will be a maximum? It is apparent that if $r$ varies from $r_n$ to $R_n$ a transition from the maximum packing to the minimum covering is obtained.

The maximum packing can be obtained if small circles are put on the surface of the sphere, then the radius of the circles is increased until the system of circles tightens and no motion of the circles is possible. This final stage of improving the arrangement is arrived at if the graph of



(a) (b)

Fig.1

the circle packing becomes infinitesimally rigid (the graph in this stage can be considered as an infinitesimally rigid bar-and-joint structure in a state of self-stress with compression members). The graph is defined as follows: the vertices of the graph are the centres of the spherical circles, the edges of the graph are the shorter great circle arcs joining the centres of two touching circles. The graph of the best packing of 24 equal circles is shown in Fig. 1(a).

The minimum covering can be obtained if large enough circles are put on the surface of the sphere, then the radius of the circles is decreased until further decrease would cause appearance of interstices, and no motion of the circles is possible. This final stage of improving the arrangement is arrived at if the graph of the circle covering becomes rigid (the graph in this stage can be considered as a rigid cable net in a state of self-stress with tensional members). The graph is a bipartite graph which is defined as follows: The vertices of the first class are the centres of the spherical circles, the vertices of the second class are the points at which three circle lines intersect, edges are the shorter great circle arcs joining first and second class vertices in a circle. The graph of the conjectured best covering of the sphere by 32 equal circles is shown in Fig. 1(b).

As $n$ is increased the densest packings become more unsymmetrical (Kottwitz, 1991). This cannot be detected for covering (Tarnai & Gáspár, 1991). Apart from the regular cases the packing and covering configurations for a given $n$ are different, and very often have different symmetries. In the intermediate problem, that is, in the transition from maximum packing to minimum covering there is a change in symmetry. Very often there are several changes in symmetry in the packing $\rightarrow$ covering transition. In the case of $n = 10$, for instance, there are five symmetry changes in the transition (Fig. 2) (Fowler & Tarnai, 1995). In Fig. 2 there are

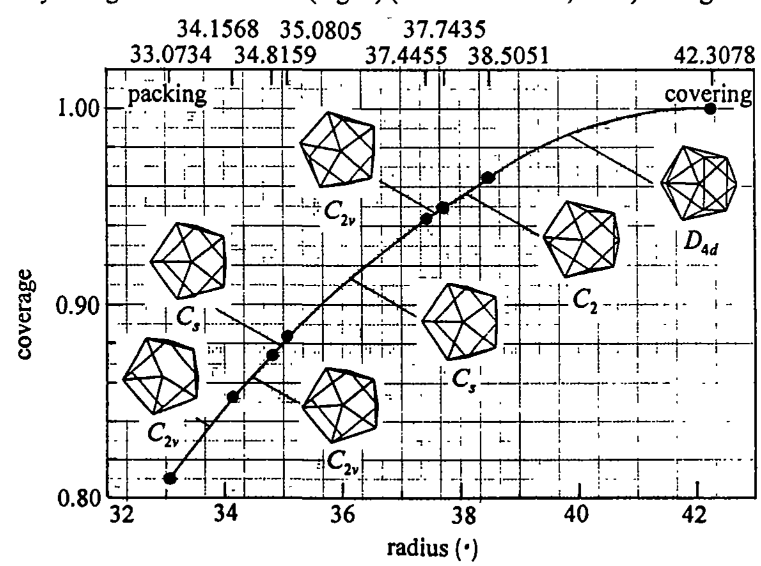
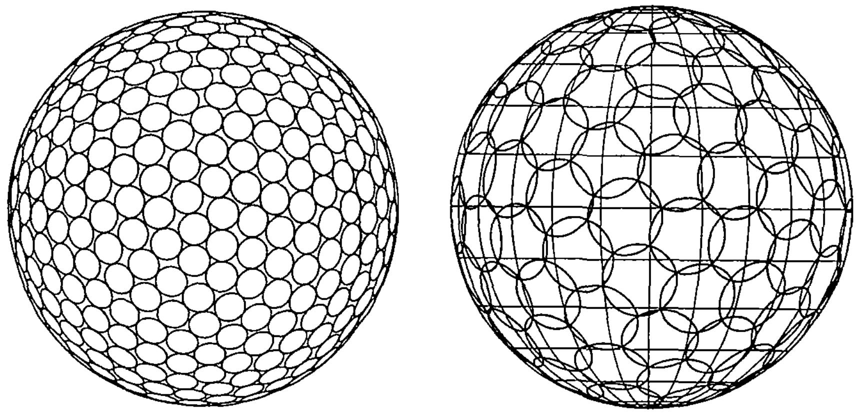



Fig. 2



(a) (b)

Fig. 3

shown contact polyhedra of the circle arrangements where an edge represents two circles having at least one point in common. Coverage is defined as the ratio of the area of the part of the sphere covered by circles to the total area of the spherical surface. Contrary to packings and coverings, for intermediate circle systems we cannot define a useful graph.

The packing and covering problem can be posed under subsidiary conditions. It is very common to use symmetry constraints. For example, we can seek for optimal packings and coverings with icosahedral symmetry. The conjectured solution to the icosahedral packing problem for $n = 492$ circles is shown in Fig. 3(a). The conjectured solution to the icosahedral covering problem for $n = 132$ circles is shown in Fig. 3(b).

These abstract mathematical problems and their solutions have many applications and examples in nature (Tarnai, 1984). It turns out that the topology and often the symmetry of known and conjectured solutions to the packing problem corresponds with chemical and biological situations, like some boranes, distribution of pores on spherical pollen grains, bunches of nuts on a walnut tree. Analogies can be established, for instance: pattern formation in foraminifers, radiolarians, sponges, mould fungi, viruses. However, the solutions of the packing problems have applications in the information theory. There is a correspondence between the topology and symmetry of arrangements found in solutions and conjectured solutions of the covering problem and many distinct physical, chemical and biological structural problems, for example: boron hydrides, carbonyls, fullerenes, certain metal alloys, clathrin cages of coated vesicles, brochosomes, coccoliths, cones of some cupressus species, soap film cones and bubbles in foam, cavities in sphere packing, and others. The solutions of the intermediate problem can represent some dynamical processes in chemistry: Berry pseudorotation, Lipscomb diamond-square-diamond rearrangement, and can correspond to the equilibrium configurations of equal point charges on the spherical surface.

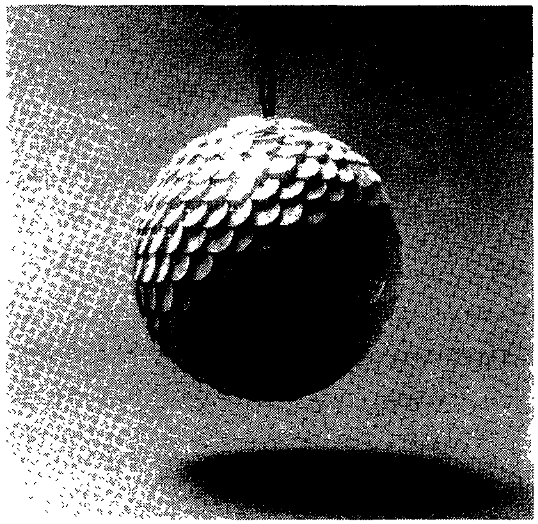


Fig. 4

Under icosahedral constraint the circle arrangements are in close relation with the structure of geodesic domes (Tarnai, 1995). The connection between viruses and geodesic domes is well known. This is an important example where symmetry can make 'bridge' between natural and artificial objects. Spherical circle packings and coverings have other applications also in the man-made world. Here we show only a christmas tree decoration (Fig. 4) as an example of a helical circle covering

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